18.01 SPRING 2005
PROBLEM SET 5
DUE THURSDAY, MARCH 17

Note: *this is due on Thursday, for a change.*

The Usual Instructions

- Write up your solutions neatly, preferably with all pages stapled. You need not show every arithmetic calculation, but must always show enough work to demonstrate the process by which the answer is reached. Without this, the grader can't be sure that you didn't just copy the answers from someone else, and there's no way to give partial credit.
- You're free to work together in groups, but you must write up the solutions independently. Plagiarism is easy to detect.
- You can either hand in your solutions in class by the due date, or slip them through the slot in my office door (2-172) by 11:59pm that night.

Reading

Notes PI, FT, Simmons 5.3, 7.1–7.2, 10.1–10.2, review 6.6–6.7.

Ungraded problems

Do the following exercises for practice—preferably after the corresponding lecture—but *do not hand them in*. The solutions are available to you, so you should check your work. Starred problems are especially recommended.

Each problem is from the Notes unless stated otherwise:

- *Tu 3/08/05:* 3D-1, 3D-3*, 3D-4*, 3D-5, 3D-7*
- *Th 3/10/05:* 3A-2, 3E-2, 3E-3*, 3E-5*; Simmons 6.7 #13
- *Fr 3/11/05:* 3E-6*, 3E-7, 4A-1, 4A-2, 4A-3, 4A-4

Graded problems, Part A [62 pts total]

From Simmons:

- 5.3 #8, 14, 22, 40, 42 [2 pts each], 62 [3 pts], 68 [4 pts]
- 6.7 #14 [4 pts], 16 [15 pts]
- 7.2 #2, 6, 10 [4 pts each]
- 10.2 #2, 4, 8, 12, 16, 40, 44 [2 pts each]

Graded problems, Part B [58 pts total]

1. [16 pts] Each of the following appeared already as an *indefinite* integral in the problems from Part A, Section 10.2. Now compute each *definite* integral by two methods:
   (i) By plugging in the limits of integration for the antiderivatives you've already computed.
   (ii) As in Simmons page 339 (expecially Example 6): by substituting to change the definite integral into a simpler definite integral with respect to some new variable u, *with different limits of integration.*

   (a) $\displaystyle\int_1^2 \frac{2x\ dx}{(4x^2-1)^2}$ (b) $\displaystyle\int_0^{\pi/2} \cos x\ e^{\sin x}\ dx$

   (c) $\displaystyle\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)\ dx$ (d) $\displaystyle\int_0^2 6x^2 e^{-x^3}\ dx$

2. [4 pts] The following simple computation is correct:

$$\int_1^2 \frac{2x\ dx}{1+x^2} = \ln(1+x^2)\big|_1^2 = \ln 5 - \ln 2 = \ln\left(\frac{5}{2}\right).$$

On the other hand, if we substitute $u = 1 + x^2$, $du = 2x\ dx$, we seem to obtain

$$\int_1^2 \frac{2x\ dx}{1+x^2} = \int_1^2 \frac{du}{u} = \ln u|_1^2 = \ln 2 - \ln 1 = \ln 2,$$

a different answer. What's wrong with this second calculation?

3. Compute $\displaystyle\int_{-2}^{2} (1 - |x-1|)\ dx$ in two ways:

   (a) [4 pts] By geometry, interpreting the integral in terms of the areas of some triangles.

   (b) [4 pts] Using the fundamental theorm of calculus. It may help to divide the interval $[-2, 2]$ into two subintervals.

   Hint: on certain subintervals, the function can be expressed without any absolute value symbols—though not in the same way on every interval.

4. (a) [4 pts] Without trying to compute the integral, show that $\displaystyle\int_{-1}^{1} \frac{dx}{x^2} > 1$.

   (b) [4 pts] Find an antiderivative of $1/x^2$. What answer do you get if you use this and the fundamental theorem of calculus to compute $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$? Notice that it contradicts part (a); in fact, it's wrong. Why does the fundamental theorem seem to fail in this case?

5. Define a function $F(x)$ by the definite integral

$$F(x) = \int_0^x \frac{1-t^2}{1+t^2} dt.$$

   You can see that $F(0) = 0$, but computing this integral for more general x is rather hard—it certainly can't be done by the methods we've covered so far, and you should not attempt it at this stage. Instead, you can learn enough about the *derivatives* of $F(x)$ to sketch a reasonable graph of the function. *Please note:* when we refer to $F(x)$ in the following, we always mean the function defined by the integral above; *this is not to be confused with the function* $\frac{1-t^2}{1+t^2}$*, which is something else!*

   (a) [2 pts] What is the domain of $F(x)$?

   (b) [4 pts] Find all critical points of $F(x)$, and determine on what intervals the function is increasing or decreasing.

   (c) [4 pts] Find all inflection points of $F(x)$, and determine on what intervals it is concave up or down.

   (d) [3 pts] How does $F'(x)$ behave as $x \to \infty$ or $x \to -\infty$?

   (e) [5 pts] Show that $F(x)$ is an *odd* function, i.e. $F(-x) = -F(x)$ for all x. *Hint:* choose an appropriate substitution to change the limits of integration.

   (f) [4 pts] Use all of this information to sketch the graph of $F(x)$. Don't worry about the precise y-coordinates of the critical points or inflection points, though you should at least be able to say whether each is above or below the x-axis.