18.01 SPRING 2005
PROBLEM SET 6
SOLUTIONS

Graded problems, Part A

See attached photocopies.

Graded problems, Part B

1. Following the hint we choose three of the tetradedron's four vertices to be the points

$$(a,0,0), \qquad \left(-\frac{1}{2}a, \frac{\sqrt{3}}{2}a, 0\right) \qquad \text{and} \qquad \left(-\frac{1}{2}a, -\frac{\sqrt{3}}{2}a, 0\right)$$

in the xy-plane, where a is a positive number. Take a moment to convince yourself that these are indeed the vertices of an equilateral triangle, and they're all the same distance from the origin. This last fact implies that the fourth vertex of the tetrahedron is on the z-axis: its coordinates are $(0,0,h)$ for some number h, which we may as well assume to be positive. We can determine h by computing the side lengths. Note first that the distance between any two of the first three vertices is

$$s = \sqrt{3}a.$$

This should be the same as the distance between $(a,0,0)$ and $(0,0,h)$, which is $\sqrt{a^2+h^2}$, thus we solve for h and find

$$h = \sqrt{2}a = \sqrt{\frac{2}{3}}s.$$

We now set up an integral for the volume by taking cross sections parallel to the xy-plane at different z-coodinates, with z varying from 0 to h. Each such cross section is an equilateral triangle, whose side length $\ell(z)$ depends linearly on z. We can determine this dependence by observing that $\ell(h) = 0$ and $\ell(0) = s$, so the unique linear function with these values is

$$\ell(z) = s\left(1 - \frac{z}{h}\right).$$

The area of each cross section is then $A(z) = \frac{\sqrt{3}}{4}[\ell(z)]^2 = \frac{\sqrt{3}}{4}s^2\left(1-\frac{z}{h}\right)^2$. Thus using these cross sections to split up the tetrahedron into slices of thickness Δz gives volume increments

$$\Delta V = \frac{\sqrt{3}}{4}s^2\left(1-\frac{z}{h}\right)^2 \Delta z,$$

and these add up to the integral

$$V = \int_0^h \frac{\sqrt{3}}{4}s^2\left(1-\frac{z}{h}\right)^2 \, dz.$$

Let's simplify this slightly by substituting $u = 1 - z/h$, so $du = -dz/h$, and the integral becomes

$$-h\int_1^0 \frac{\sqrt{3}}{4}s^2u^2 \, du = \frac{\sqrt{3}}{4}hs^2\int_0^1 u^2 \, du = \frac{\sqrt{3}}{4}hs^2 \left.\frac{u^3}{3}\right|_0^1 = \frac{1}{4\sqrt{3}}hs^2 = \frac{1}{4\sqrt{3}}\frac{\sqrt{2}}{\sqrt{3}}s^3 = \frac{\sqrt{2}}{12}s^3.$$

Note that both this and the formula of Problem (4B-3) in the Notes are special cases of a general fact: take any polygon with area A in the plane and extend it in three dimensions to a pyramid of height h. Then the volume of the pyramid is always

$$V = \frac{1}{3}Ah.$$

In the case of the tetrahedron we have $A = \frac{\sqrt{3}}{4}s^2$ and $h = \sqrt{\frac{2}{3}}s$, thus reproducing the formula above. The general case can be proved by a similar argument—you only have to observe that the cross sectional areas depend *quadratically* on z.