18.01 SPRING 2005
PROBLEM SET 7
SOLUTIONS

Graded problems, Part A

See attached photocopies.

Graded problems, Part B

1. (a) We can find constants A and B such that

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

for all x: it's easy to check that this works if $A = 1/2$ and $B = -1/2$. Thus

$$\int \frac{dx}{x^2-1} = \int \left(\frac{1/2}{x-1} - \frac{1/2}{x+1} \right) dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = \boxed{\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}.$$

(b) The appropriate substitution is $x = \sec\theta$, since then $x^2 - 1$ becomes $\sec^2\theta - 1 = \tan^2\theta$. We have now $dx = \sec\theta\tan\theta\, d\theta$, so

$$\int \frac{dx}{x^2-1} = \int \frac{\sec\theta\tan\theta\, d\theta}{\tan^2\theta} = \int \frac{\sec\theta}{\tan\theta} d\theta = \int \frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta} d\theta = \int \csc\theta\, d\theta = -\ln|\csc\theta + \cot\theta|$$

using the formula from the inside back cover of Simmons. (The absolute value here and in part (a) is a technicality—we can't take logarithms of negative numbers, yet one wants the expression to make sense even if negative numbers pop up, and taking the absolute value gives a correct result. In practice this issue rarely arises and is hardly worth thinking about.) To rewrite the answer as a function of x, we can either use the right triangle trick for each of $\csc\theta$ and $\cot\theta$, or simply observe

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{x^2-1}}$$

due to the identity we used above, and also

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\sqrt{1-\cos^2\theta}} = \frac{1}{\sqrt{1-\frac{1}{x^2}}}$$

since $1/x = \cos\theta$. Multiplying the top and bottom of this expression by x gives $\dfrac{x}{\sqrt{x^2-1}}$, and we then have the answer

$$-\ln\left|\frac{x}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}}\right| = \boxed{-\ln\left|\frac{x+1}{\sqrt{x^2-1}}\right|}.$$

Plus a constant, of course.

(c) It happens that the two answers we've obtained *are* equal—we can turn the second one into the first one as follows:

$$-\ln\left|\frac{x+1}{\sqrt{x^2-1}}\right| = -\ln\left|\frac{x+1}{\sqrt{(x-1)(x+1)}}\right| = -\ln\left|\frac{\sqrt{x+1}}{\sqrt{x-1}}\right| = \ln\left|\sqrt{\frac{x-1}{x+1}}\right| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|.$$

This should be a clue that we're doing something right.

2. The antiderivative calculation is fine, but one most note that neither the original function nor the antiderivative is defined for $-1 \leq x \leq 1$—attempting to compute them gives a delicious combination of division by zero and square roots of negative numbers. Integrating such a function from -2 to 2 makes no sense at all, and the "answer" obtained by computing the antiderivative is irrelevant, i.e. it has nothing to do with the area under any curve. (Contrast this with the case of $\int_{-1}^{1} \frac{dx}{x^2}$, where the function is undefined at a single point, but if you draw the graph you can still see what "area under the curve" means. In the present case, there is simply no curve to compute the area under as x moves through the interval $[-1, 1]$.) On the other hand, one could use the same method to compute something like

$$\int_{2}^{4} \frac{dx}{\sqrt{x^2 - 1}}$$

and there would be no problem.