18.01 SPRING 2005
PROBLEM SET 9
SOLUTIONS

Graded problems, Part A

See attached photocopies.

Graded problems, Part B

1. (a) Let's first note that the point $(0, 0)$ is clearly part of the curve $x^3 + y^3 = 3axy$, and in fact it's the only such point with $x = 0$. With that understood, we assume $x \neq 0$ and set $t = y/x$. Plugging $y = tx$ into the equation above gives

$$x^3 + t^3x^3 = 3atx^2 \Longrightarrow x(1 + t^3) = 3at,$$

thus for $t \neq -1$ we can divide by $1 + t^3$, obtaining

$$x = \frac{3at}{1+t^3} \quad \text{and} \quad y = tx = \frac{3at^2}{1+t^3}.$$

When $t = -1$ the equation $x(1 + t^3) = 3at$ has no solution, so we'll have to exclude this from the domain of parametrization. Notice however that the point $(0, 0)$ *is* covered by this parametrization after all: $(x, y) = (0, 0)$ when $t = 0$.

(b) As $t \to -1$,

$$x(t) + y(t) = \frac{3at}{1+t^3} + \frac{3at^2}{1+t^3} = \frac{3at(1+t)}{(1+t)(1-t+t^2)} = \frac{3at}{1-t+t^2} \to \frac{-3a}{1+1+1} = -a.$$

This shows that as $t \to -1$ (from either side), the points $(x(t), y(t))$ approach the line $x+y = -1$. To see that the result is an asymptote, we only need observe that $x(t)$ and $y(t)$ both become *infinite* as $t \to -1$.

(c) $\dfrac{dx}{dt} = \dfrac{3a(1+t^3) - 3at(3t^2)}{(1+t^3)^2} = \dfrac{3a - 6at^3}{(1+t^3)^2} = 0$ if and only if $3a - 6at^3 = 3a(1 - 2t^3) = 0 \Rightarrow t = 1/\sqrt[3]{2}$. Likewise $\dfrac{dy}{dt} = \dfrac{6at(1+t^3) - 3at^2(3t^2)}{(1+t^3)^2} = \dfrac{6at - 3at^4}{(1+t^3)^2} = 0$ if $6at - 3at^4 = 3at(2 - t^3) = 0 \Rightarrow t = 0$ or $\sqrt[3]{2}$. Notice that there are no points where *both* derivatives vanish. The tangent line is thus vertical at

$$t = \frac{1}{\sqrt[3]{2}} \qquad \Longrightarrow \qquad (x, y) = (2^{2/3}a, 2^{1/3}a),$$

and horizontal at

$$t = 0 \qquad \Longrightarrow \qquad (x, y) = (0, 0)$$

and

$$t = \sqrt[3]{2} \qquad \Longrightarrow \qquad (x, y) = (2^{1/3}a, 2^{2/3}a).$$

(d) To show that the curve is symmetric about the diagonal line $y = x$, we must see that if (x, y) is any point on the curve, then (y, x) is also on the curve. This is obvious from the equation $x^3 + y^3 = 3axy$, since it is unchanged if we reverse the roles of x and y.

(e) As $t \to \pm\infty$, x and y both approach 0. We therefore begin drawing the curve from the origin. As t moves up from $-\infty$ and approaches -1, $x \to +\infty$ and $y \to -\infty$, thus the curve moves down toward the right and approaches the asymptote $x + y = -1$ in the fourth quadrant. When t is slightly greater than -1, x and y are again large but have opposite signs, so the curve moves down from the asymptote in the second quadrant. At $t = 0$ it passes through the origin with a horizontal slope. It then becomes vertical at $t = 1/\sqrt[3]{2}$, passing upward through $(2^{2/3}a, 2^{1/3}a)$, and becomes horizontal again at $t = \sqrt[3]{2}$, $(x, y) = (2^{1/3}a, 2^{2/3}a)$. There are no more horizontal or vertical points as $t \to \infty$, so we complete the trajectory by curving down and left back to the origin. Notice that the picture is symmetric about the diagonal line $y = x$.

Notice also that the curve appears to intersect itself in a right angle at the origin, though technically the portion passing through vertically is two separate legs of the parametric curve (the two portions traced as $t \to -\infty$ and $+\infty$). We haven't explicitly verified that the curve should have a vertical slope here, but it follows from the symmetry since we *do* know that the other part of this intersection passes through horizontally. We could check the vertical slope more directly by writing down the slope as a function of t:

$$m = \frac{dy/dt}{dx/dt} = \frac{t(2 - t^3)}{1 - 2t^3}.$$

The result follows because this slope becomes infinite as $t \to \pm\infty$.