18.950 SPRING 2007
PROBLEM SET 3
DUE FRIDAY, MARCH 9, 1:00PM

You can hand in the problem set in class or at my office (2-169) anytime before it's due. (Slip it through the slot in the door if I'm not there.)

Reading

Spivak: Chapter 4.

By this point Spivak is using the terminology of vector bundles quite extensively—a serious topic that we will get to soon, but not quite yet. So in reading Chapter 4, for most purposes you can assume the vector bundle that Spivak usually denotes by ξ is simply the tangent bundle TM, and its "dual bundle" ξ^* is the cotangent bundle T^*M.

A word on definitions and notation

Spivak's notation for spaces and types of tensors differs significantly from what we've used in lecture, so let's quickly codify our notation. If V is a vector space, we define V_ℓ^k to be the vector space of tensors on V that are *covariant* of rank ℓ and *contravariant* of rank k, i.e. multilinear maps

$$\underbrace{V \times \ldots \times V}_{\ell} \times \underbrace{V^* \times \ldots \times V^*}_{k} \to \mathbb{R}$$

where we denote the dual space $V^* = \mathrm{Hom}(V, \mathbb{R}) = V_1^0$. Replacing V with a tangent space T_pM, we call the corresponding tensor space $(T_\ell^k M)_p$ and define the *tensor bundle* $T_\ell^k M$ to be the union of these spaces for all $p \in M$. A *tensor of type* (k, ℓ) on T_pM is then simply an element of $(T_\ell^k M)_p$, and a *tensor field*[1] of type (k, ℓ) smoothly assigns to each $p \in M$ an element of $(T_\ell^k M)_p$. The space of tensor fields of this type is denoted $\Gamma(T_\ell^k M)$, literally, "sections of the bundle $T_\ell^k M$". We'll define precisely what *section* means when we discuss bundles in earnest.

Here's a brief glossary of some notational differences between our discussion and Spivak's. Each choice has its own logic, and neither is perfect.

our notation		**Spivak's notation**
$V_k^0, T_k^0 M, (T_k^0 M)_p$	$=$	$\mathcal{T}^k(V), \mathcal{T}^k(TM), \mathcal{T}^k(M_p)$
$V_0^k, T_0^k M, (T_0^k M)_p$	$=$	$\mathcal{T}_k(V), \mathcal{T}_k(TM), \mathcal{T}_k(M_p)$
$V_\ell^k, T_\ell^k M, (T_\ell^k M)_p$	$=$	$\mathcal{T}_k^\ell(V), \mathcal{T}_k^\ell(TM), \mathcal{T}_k^\ell(M_p)$
tensor of type (k, ℓ)	$=$	tensor of type $\binom{\ell}{k}$

Yes, that's right, the k and ℓ are flipped—I swear it's not my fault. My logic (and I'm not the only one) is that the *covariant* tensors, i.e. those which act on vectors but not on dual vectors, should be indicated by a *lower* index because their components have lower indices. Similarly it makes sense to say $TM = T_0^1 M$ instead of $T_1^0 M$ because the components of tangent vectors have upper indices. Thankfully, Spivak uses the words *covariant* and *contravariant* the same way we do (though we will not use them often), and his notation for indices of components is completely consistent with ours.

[1] One often omits the word "field" from "tensor field" when there's no danger of confusion.

Problems [52 pts total]

1. [5 points] Let V and W be vector spaces, and for $k \in \mathbb{N}$ denote by $\mathrm{Hom}_k(V, W)$ the vector space of k-multilinear maps $\underbrace{V \times \ldots \times V}_{k} \to W$. Find a natural isomorphism $\mathrm{Hom}_k(V, V) \to V^1_k$, and prove that it is an isomorphism. *Note:* there are multiple isomorphisms that one could write down, but only one that is truly *natural*. To prove it's an isomorphism, remember it suffices to show that the map is linear and injective, and that $\dim \mathrm{Hom}_k(V, V) = V^1_k$. What are these dimensions actually? If it simplifies things, you may as well assume $V = \mathbb{R}^n$.

2. Let M be an n-dimensional manifold with an open set $\mathcal{U} \subset M$ and coordinate chart $x = (x^1, \ldots, x^n) : \mathcal{U} \to \mathbb{R}^n$. Recall that the coordinate functions $x^j : \mathcal{U} \to \mathbb{R}$ define derivations $\partial_j := \frac{\partial}{\partial x^j}$ and differentials dx^j, which give bases of T_pM and T^*_pM respectively at every point $p \in \mathcal{U}$. With these, an arbitrary tensor field $T \in \Gamma(T^k_\ell M)$ can be expressed over $\mathcal{U}$ via its $n^{k+\ell}$ *component functions* $T^{i_1 \ldots i_k}{}_{j_1 \ldots j_\ell} : \mathcal{U} \to \mathbb{R}$, defined by
$$T^{i_1 \ldots i_k}{}_{j_1 \ldots j_\ell} = T(\partial_{j_1}, \ldots, \partial_{j_\ell}, dx^{i_1}, \ldots, dx^{i_k}).$$
We then have
$$T = T^{i_1 \ldots i_k}{}_{j_1 \ldots j_\ell}\; dx^{j_1} \otimes \ldots \otimes dx^{j_\ell} \otimes \partial_{i_1} \otimes \ldots \otimes \partial_{i_k},$$
using the Einstein summation convention: recall that since this expression contains $k + \ell$ pairs of matching upper and lower indices, there's an implied summation over each one. Literally then (we'll write it out just this once), this means
$$T = \sum_{i_1=1}^{n} \ldots \sum_{i_k=1}^{n} \sum_{j_1=1}^{n} \ldots \sum_{j_\ell=1}^{n} T^{i_1 \ldots i_k}{}_{j_1 \ldots j_\ell}\; dx^{j_1} \otimes \ldots \otimes dx^{j_\ell} \otimes \partial_{i_1} \otimes \ldots \otimes \partial_{i_k}.$$
That's why we usually don't write it out literally.

   (a) [5 points] Consider a tensor field S of type $(3, 2)$ and another T of type $(2, 1)$, and recall that the *tensor product* $S \otimes T$ is then a tensor field of type $(5, 3)$ defined by
   $$(S \otimes T)(X, Y, Z, \alpha, \beta, \gamma, \theta, \omega) = S(X, Y, \alpha, \beta, \gamma) \cdot T(Z, \theta, \omega)$$
   for any tangent vectors $X, Y, Z \in T_pM$ and cotangent vectors $\alpha, \beta, \gamma, \theta, \omega \in T^*_pM$. Find a formula for the component functions $(S \otimes T)^{ijk\ell m}{}_{pqr} : \mathcal{U} \to \mathbb{R}$ in terms of $S^{ijk}{}_{pq}$ and $T^{\ell m}{}_r$. The answer is quite simple—and though we've chosen tensors of relatively low rank to simplify the notation, you can see what the answer for tensors of general type would be.

Now suppose $\hat{x} = (\hat{x}^1, \ldots, \hat{x}^n) : \widehat{\mathcal{U}} \to \mathbb{R}^n$ is another coordinate chart on some open subset $\widehat{\mathcal{U}} \subset M$ such that $\mathcal{U} \cap \widehat{\mathcal{U}} \neq \emptyset$. Denote by $\widehat{T}^{i_1 \ldots i_k}{}_{j_1 \ldots j_\ell} : \widehat{\mathcal{U}} \to \mathbb{R}$ the component functions for a tensor field $T \in \Gamma(T^k_\ell M)$ in the new chart. As we mentioned in Problem Set 2, the basis vectors $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial \hat{x}^j}$ in T_pM for any point $p \in \mathcal{U} \cap \widehat{\mathcal{U}}$ are related to each other by
$$\frac{\partial}{\partial x^j} = \frac{\partial \hat{x}^i}{\partial x^j} \frac{\partial}{\partial \hat{x}^i}, \tag{1}$$
where this time we're using the summation convention to imply a summation over the repeated index i (it's considered a *lower* index in $\frac{\partial}{\partial \hat{x}^i}$ because it appears in the denominator). The partial derivatives $\frac{\partial \hat{x}^i}{\partial x^j}$ for each i and j should best be understood as smooth functions $\mathcal{U} \cap \widehat{\mathcal{U}} \to \mathbb{R}$, though of course we'd have to use the coordinates and express them as functions on the open set $x(\mathcal{U} \cap \widehat{\mathcal{U}}) \subset \mathbb{R}^n$ in order to compute them. Let us be more explicit: denote by $\left.\frac{\partial}{\partial x^j}\right|_p$ the actual vector in T_pM which is the value of the coordinate vector field $\frac{\partial}{\partial x^j} \in \mathrm{Vec}(\mathcal{U})$ at $p \in \mathcal{U}$, and define $\left.\frac{\partial}{\partial \hat{x}^j}\right|_p$ similarly for $p \in \widehat{\mathcal{U}}$. Then Equation (1) says that for all $p \in \mathcal{U} \cap \widehat{\mathcal{U}}$,
$$\left.\frac{\partial}{\partial x^j}\right|_p = \frac{\partial \hat{x}^i}{\partial x^j}(p)\; \left.\frac{\partial}{\partial \hat{x}^i}\right|_p.$$

(b) [3 points] Derive a similar expression for the coordinate 1-forms dx^j in terms of $d\hat{x}^j$ and $\frac{\partial x^j}{\partial \hat{x}^i}$ at points in $\mathcal{U} \cap \widehat{\mathcal{U}}$. *This is quite easy: just remember dx^j is the differential of a smooth function!*

(c) [5 points] For a 1-form $\lambda \in \Gamma(T_1^0)$ and a covariant rank 2 tensor field $T \in \Gamma(T_2^0 M)$, use the above relations between dx^j and $d\hat{x}^j$ to derive the transformation formulas

$$\hat{\lambda}_i = \lambda_j \frac{\partial x^j}{\partial \hat{x}^i} \quad \text{and} \quad \widehat{T}_{ij} = T_{k\ell} \frac{\partial x^k}{\partial \hat{x}^i} \frac{\partial x^\ell}{\partial \hat{x}^j},$$

relating the distinct sets of component functions over $\mathcal{U} \cap \widehat{\mathcal{U}}$.

(d) [5 points] For a contravariant rank 2 tensor field $T \in \Gamma(T_0^2 M)$, derive

$$\widehat{T}^{ij} = \frac{\partial \hat{x}^i}{\partial x^k} \frac{\partial \hat{x}^j}{\partial x^\ell} T^{k\ell}.$$

(e) [5 points] Finally, for a tensor field $A \in \Gamma(T_1^1 M)$ of "mixed" type $(1,1)$, show that

$$\widehat{A}^i{}_j = \frac{\partial \hat{x}^i}{\partial x_k} A^k{}_\ell \frac{\partial x^\ell}{\partial \hat{x}^j}.$$

This formula has a nice interpretation using matrices: define the smooth matrix-valued function $\mathbf{A} : \mathcal{U} \to \mathbb{R}^{n \times n}$ *by setting the entry at the ith row and jth column of $\mathbf{A}(p)$ to $A^i{}_j(p)$, and define* $\widehat{\mathbf{A}} : \widehat{\mathcal{U}} \to \mathbb{R}^{n \times n}$ *similarly. We can also define the partial derivative matrix $\mathbf{S} : \mathcal{U} \cap \widehat{\mathcal{U}} \to \mathbb{R}^{n \times n}$ with entries $\mathbf{S}^i{}_j = \frac{\partial \hat{x}^i}{\partial x^j}$, and observe that by the inverse function theorem, $\mathbf{S}^{-1}$ is the matrix with entries $\frac{\partial x^i}{\partial \hat{x}^j}$. Then the transformation formula above becomes*

$$\widehat{\mathbf{A}} = \mathbf{S}\mathbf{A}\mathbf{S}^{-1}.$$

3. Recall that in lecture we used the concept of C^∞*-linearity* to prove that for any 1-form λ, the bilinear map $T : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to \mathbb{R}$ defined by

$$T(X,Y) = L_X(\lambda(Y)) - L_Y(\lambda(X)) - \lambda([X,Y])$$

defines a tensor. Let's be clear on the meaning of this expression: X and Y are vector fields, thus $\lambda(X)$ is the smooth real valued function $p \mapsto \lambda(X(p))$, and its Lie derivative with respect to Y is another smooth real valued function, as is $\lambda([X,Y])$. Thus the entire expression defines a real valued function, and when we say it *defines a tensor*, we mean that the value of this function at p depends only on $X(p)$ and $Y(p)$, not on any extra information about X and Y as vector fields (e.g. their derivatives). This statement is nontrivial because, e.g. it's *not true* for any of the individual terms on the right hand side—but somehow their dependence on derivatives of X and Y cancels out in the sum. Clearly any bilinear map with these properties satisfies

$$T(fX,Y) = f \cdot T(X,Y) = T(X,fY)$$

for all C^∞ functions $f : M \to \mathbb{R}$, and we mentioned in lecture the important lemma (Spivak p. 118, Theorem 2), that the converse is also true: a multilinear map $T : \mathrm{Vec}(M) \times \cdots \times \mathrm{Vec}(M) \to \mathbb{R}$ defines a tensor if it is C^∞-linear in each variable.

(a) [5 points] The one detail we left out of our computation in lecture was the proof of the formula

$$[fX,Y] = f[X,Y] - (L_Y f) \cdot X \tag{2}$$

for any $X, Y \in \mathrm{Vec}(M)$ and $f \in C^\infty(M)$. Prove this.

Now take a moment to remind yourself how this is used to prove that T defines a tensor. Observe also that as a consequence of Equation (2), the bracket itself $[\ ,\ ] : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to \mathbb{R}$ does not define a tensor.

(b) [5 points] Choosing local coordinates $(x^1, \dots, x^n)$ near some point $p \in M$, show that the component functions λ_i and T_{ij} are related by

$$T_{ij} = \partial_i \lambda_j - \partial_j \lambda_i.$$

You may find it helpful to recall that brackets of coordinate vector fields always vanish, that is, $[\partial_i, \partial_j] \equiv 0$.

(c) [5 points] Show that the expression $S_{ij} = \partial_i \lambda_j$ does *not* define a tensor, i.e. there is no tensor S of type $(0, 2)$ whose component functions in local coordinates equal S_{ij}.

There are two ways you could go about this. One is to consider another coordinate chart $\hat{x}$ that overlaps with x, use the formulas of Problem (2c) to write down the transformed component functions $\widehat{S}_{ij}$ and $\hat{\lambda}_i$ and show that $\widehat{S}_{ij} = \frac{\partial}{\partial \hat{x}^i} \hat{\lambda}_j$ does not hold in general.

But if you want to be more clever about it, you could find a bilinear map $S : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to \mathbb{R}$ such that $S(\partial_i, \partial_j) = \partial_i \lambda_j$, and show that this map is not C^∞-linear. There is such a map, staring you in the face.

4. In this problem, denote the entries of an n-by-n matrix $\mathbf{A}$ by $\mathbf{A}^i{}_j$. Thus multiplication of two n-by-n matrices can be expressed using the summation convention as

$$(\mathbf{AB})^i{}_j = \mathbf{A}^i{}_k \mathbf{B}^k{}_j.$$

The *trace* of a matrix is the scalar $\mathrm{tr}(\mathbf{A})$ obtained by summing the diagonal entries: using the summation convention,

$$\mathrm{tr}(\mathbf{A}) = A^i{}_i.$$

(a) [3 points] Show that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$ for any pair of n-by-n matrices.

(b) [2 points] Use the above to show that $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{BAB}^{-1})$.

(c) [4 points] If $A \in \Gamma(T^1_1 M)$ is a tensor field of type $(1, 1)$, we define the *contraction* of A to be the smooth real valued function $\mathrm{tr}\, A : M \to \mathbb{R}$ which equals

$$\mathrm{tr}\, A = A^i{}_i,$$

where $A^i{}_j$ are the components of A in any local coordinate system. In other words, to compute $\mathrm{tr}\, A(p)$ for $p \in M$, we pick a coordinate chart on a neighborhood $\mathcal{U}$ of p, write down the corresponding component functions $A^i{}_j : \mathcal{U} \to \mathbb{R}$ and compute the above expression at p. Explain why the result is independent of the choice of chart. (See the discussion at the end of Problem 2.)

For mixed tensors of higher rank there are more general contractions that can be defined: e.g. from a tensor field of type $(4, 2)$ *with components* $T^{ijk\ell}{}_{pr}$, *one can define one of type* $(3, 1)$ *whose components are*

$$S^{ijk}{}_p := T^{ijk\ell}{}_{p\ell}\,.$$

By a slight extension of the argument for tensors of type $(1, 1)$, *such operations give well defined homomorphisms*

$$T^k_\ell M \to T^{k-1}_{\ell-1} M.$$