18.950 SPRING 2007
PROBLEM SET 4
DUE WEDNESDAY, APRIL 11, 1:00PM

It's not as long as it looks. Still, don't leave it all for the last moment.

You can hand in the problem set in class or at my office (2-169) anytime before it's due. (Slip it through the slot in the door if I'm not there.)

Reading

Lecture notes Chapter 1, §2.1 and 2.2.

Problems [65 pts total]

1. Let's have some fun with Leibnitz rules (and tie up a loose end regarding differential forms while we're at it). Recall that the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfies the "graded Leibnitz rule"

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

if $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$. Such a formula is also satisfied by the *interior product*: recall that for a vector field $X \in \mathrm{Vec}(M)$ and k-form ω, we define a $(k-1)$-form $\iota_X\omega(Y_1, \ldots, Y_{k-1}) := \omega(X, Y_1, \ldots, Y_{k-1})$. Extend this definition to 0-forms $f \in \Omega^0(M) = C^\infty(M)$ by setting $\iota_X f := 0$. Now a (sadly) not very illuminating computation in Spivak (p. 227, Problem 4) proves that ι_X satisfies another graded Leibnitz rule,

$$\iota_X(\alpha \wedge \beta) = \iota_X\alpha \wedge \beta + (-1)^k \alpha \wedge \iota_X\beta$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$.

Recall from Midterm 1 that the *Lie derivative* of a form $\omega \in \Omega^k(M)$ with respect to $X \in \mathrm{Vec}(M)$ is defined by

$$L_X\omega = \frac{d}{dt}(\varphi_X^t)^*\omega\bigg|_{t=0},$$

where φ_X^t is the flow of X, and for diffeomorphisms $\psi : M \to M$ in general the pullback $\psi^*\omega$ is the form in $\Omega^k(M)$ defined by

$$\psi^*\omega(Y_1, \ldots, Y_k) = \omega(T\psi(Y), \ldots, T\psi(Y)).$$

It's not too hard to show that both the wedge product and the exterior derivative are well behaved with respect to pullbacks: i.e.

$$\psi^*(\alpha \wedge \beta) = \psi^*\alpha \wedge \psi^*\beta \quad \text{and} \quad \psi^*(d\omega) = d(\psi^*\omega). \tag{1}$$

These are also valid for 0-forms $f \in \Omega^0(M) = C^\infty(M)$, where by definition $\psi^* f := f \circ \psi$. The relations are proven in Spivak; take them on faith for now if you feel skepticle. Using the first in particular, along with the ordinary product rule from single variable calculus, we obtain a Leibnitz rule for the Lie derivative:

$$L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta.$$

Notice the lack of any irritating sign: this has something to do with the fact that L_X is an operator "of degree 0", i.e. it takes forms of degree k to forms of degree k, whereas d and ι_X each map to forms of one degree higher or lower. In this sense, L_X is an "even" operator, where d and ι_X are both "odd": the sign in the Leibnitz formula can be thought of as appearing whenever we exchange the order of an odd operator and a form of odd degree.

(a) [5 pts] Given $X \in \mathrm{Vec}(M)$, define another operator $P_X : \Omega^k(M) \to \Omega^k(M)$ by

$$P_X = d \circ \iota_X + \iota_X \circ d.$$

Use the two graded Leibnitz rules above to show that P_X also satisfies

$$P_X(\alpha \wedge \beta) = P_X\alpha \wedge \beta + \alpha \wedge P_X\beta.$$

(b) [4 pts] Show that for all $f \in \Omega^0(M)$, $L_X f = P_X f$.

(c) [10 pts] Show that for all $f \in \Omega^0(M)$, $L_X df = P_X df$. *Hint:* this is a little tricky, but straightforward if you keep in mind that every tangent vector on M is also a velocity vector of some smooth path. And of course, $\frac{\partial}{\partial s}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}$.

(d) [4 pts] It follows from all these results that

$$L_X = d \circ \iota_X + \iota_X \circ d \tag{2}$$

on all differential forms. Why?

2. (a) [5 pts] For the 2-dimensional torus $T^2 = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$, show that its tangent bundle $TT^2 \to T^2$ is (globally) trivializable.

(b) [5 pts] Let S^1 be the unit circle in $\mathbb{C}$ and define a real line bundle $\ell \to S^1$ by

$$\ell = \bigcup_{\theta \in \mathbb{R}} \{e^{i\theta}\} \times \ell_\theta$$

where ℓ_θ is the real 1-dimensional subspace

$$\ell_\theta = \mathbb{R}\begin{pmatrix} \cos(k\theta/2) \\ \sin(k\theta/2) \end{pmatrix} \subset \mathbb{R}^2$$

for some $k \in \mathbb{Z}$. If k is even, find explicitly a global trivialization of ℓ.

We saw in class that ℓ is not *trivializable if $k = 1$, and in fact this is true for any k odd. Don't prove this, but think about it. See also Example 2.12 in the notes.*

3. [10 pts] Show that the projective plane $\mathbb{R}P^2 = S^2/\mathbb{Z}_2$ is not orientable. Do this by finding a continuous path $\gamma : [0,1] \to \mathbb{R}P^2$ with $\gamma(0) = \gamma(1)$ and a continuous family of bases of the tangent spaces $T_{\gamma(t)}\mathbb{R}P^2$, such that the bases at $\gamma(0)$ and $\gamma(1)$ cannot be deformed into one another. *Hint:* recall that $\mathbb{R}P^2$ can be visualized as

$$\mathbb{D}/\sim$$

where $\mathbb{D}$ is the closed unit disk in $\mathbb{R}^2$ and we identify opposite points on the boundary by an equivalence relation: $\mathbf{x} \sim -\mathbf{x}$ for all $\mathbf{x} \in \partial\mathbb{D}$. You need not write everything down in explicit formulas—for the most part, some pictures and a little explanation will suffice.

4. Never mind the bundles, let's talk about forms again. Now that we've proved the formula (2), it would be a shame not to use it for something. Here is a beautiful application to Hamiltonian dynamics.

We begin with a little motivation. In the classical mechanics of Newton, the motion of a system with n degrees of freedom is described by n time-dependent position variables $q_1(t), \ldots, q_n(t)$ with associated masses $m_1, \ldots, m_n \geq 0$, and their accelerations $\ddot{q}_j(t)$ satisfy $F_j = m_j\ddot{q}_j$.[1] Write $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n$ and $\mathbf{F} = (F_1, \ldots, F_n) \in \mathbb{R}^n$. The force vector often depends on the n position variables and is given as minus the gradient of a smooth *potential* function $V : \mathbb{R}^n \to \mathbb{R}$:

$$\mathbf{F} = -\nabla V(\mathbf{q}).$$

[1]Just this once, we're ignoring the usual superscript/subscript index conventions; it's more trouble than it's worth.

Then $\mathbf{q}(t)$ satisfies the system of n second order differential equations

$$\ddot{q}_j = -\frac{1}{m_j}\frac{\partial V}{\partial q_j}(\mathbf{q}).$$

The motion of $\mathbf{q}(t)$ is uniquely determined by these equations together with the initial positions $\mathbf{q}(0)$ and velocities $\dot{\mathbf{q}}(0)$.

Hamiltonian mechanics reformulates this as follows. The total energy of the system at any given time is obtained by adding to $V(\mathbf{q})$ a *kinetic energy* term, thus

$$E = \sum_j \frac{1}{2} m_j \dot{q}_j^2 + V(\mathbf{q}).$$

Now, each coordinate q_j has a corresponding *momentum* $p_j = m_j \dot{q}_j$ which gives another time-dependent path $\mathbf{p}(t) = (p_1(t), \ldots, p_n(t)) \in \mathbb{R}^n$, and the energy can be expressed as a function of $\mathbf{q}$ and $\mathbf{p}$: we call this expression the *Hamiltonian* function

$$H(\mathbf{q}, \mathbf{p}) = \sum_j \frac{p_j^2}{2m_j} + V(\mathbf{q}).$$

Now it's easy to check that the path $(\mathbf{q}(t), \mathbf{p}(t)) \in \mathbb{R}^{2n}$ satisfies the system of $2n$ first order differential equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j}(\mathbf{q}, \mathbf{p}), \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}(\mathbf{q}, \mathbf{p}). \tag{3}$$

These are called *Hamilton's equations*, and in this context the space $\mathbb{R}^{2n}$ in which $\mathbf{q}(t)$ and $\mathbf{p}(t)$ move together is called *phase space*. In this way the system of second order differential equations described by Newton can be reduced to first order equations, at the cost of having twice as many. Since the equations of motion in phase space are first order, the path $(\mathbf{q}(t), \mathbf{p}(t)) \in \mathbb{R}^{2n}$ is uniquely determined by its initial condition $(\mathbf{q}(0), \mathbf{p}(0))$, a point in phase space.

We shall now prove *Liouville's theorem*:

Theorem. *For any smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$, the motion of the system (3) in $\mathbb{R}^{2n}$ is volume preserving.*

This means the following: if $\Omega \subset \mathbb{R}^{2n}$ has volume $\mathrm{Vol}(\Omega)$, T is a real number and $\Omega_T \subset \mathbb{R}^{2n}$ is the set of all points of the form $(\mathbf{q}(T), \mathbf{p}(T))$ such that the path $(\mathbf{q}(t), \mathbf{p}(t))$ satisfies (3) and $(\mathbf{q}(0), \mathbf{p}(0)) \in \Omega$, then $\mathrm{Vol}(\Omega_T) = \mathrm{Vol}(\Omega)$. In other words, *Hamiltonian systems preserve volumes in phase space.* Note that the statement doesn't assume any knowledge of Newtonian mechanics at all: $H(\mathbf{q}, \mathbf{p})$ need not be the sum of kinetic and potential energies as we originally defined it, rather it could be any smooth function on $\mathbb{R}^{2n}$.

(a) [4 pts] Define the 2-form

$$\omega = \sum_{j=1}^n dp_j \wedge dq_j,$$

which is called the *standard symplectic form* on $\mathbb{R}^{2n}$. Show that

$$\omega^n := \underbrace{\omega \wedge \ldots \wedge \omega}_{n}$$

is a constant multiple of the volume form

$$dq_1 \wedge dp_1 \wedge \ldots \wedge dq_n \wedge dp_n.$$

Hint: recall that for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$, $\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$; the sign is negative if and only if both k and ℓ are odd. This implies in particular that $\lambda \wedge \lambda = 0$ for any 1-form λ, but this is not generally true for 2-forms.

(b) [4 pts] A diffeomorphism $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called *symplectic* if $\psi^*\omega = \omega$. Show that all such maps are volume preserving. *Hint:* this is easy if you recall some of what you learned about volume preserving maps on the midterm, and use (1).

(c) [5 pts] Define the *Hamiltonian vector field* $X_H \in \mathrm{Vec}(\mathbb{R}^{2n})$ by

$$X_H(\mathbf{q}, \mathbf{p}) = \sum_j \frac{\partial H}{\partial p_j}(\mathbf{q}, \mathbf{p}) \frac{\partial}{\partial q_j} - \sum_j \frac{\partial H}{\partial q_j}(\mathbf{q}, \mathbf{p}) \frac{\partial}{\partial p_j},$$

so if $\mathbf{x}(t) = (\mathbf{q}(t), \mathbf{p}(t)) \in \mathbb{R}^{2n}$, Hamilton's equations (3) become $\dot{\mathbf{x}} = X_H(\mathbf{x})$. Liouville's theorem is now equivalent to the statement that the flow $\varphi^t_{X_H} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is volume preserving for all t. It turns out that X_H is the unique vector field on $\mathbb{R}^{2n}$ satisfying the relation

$$dH = -\iota_{X_H}\omega. \tag{4}$$

Prove this in the case $n = 1$. *Hint:* recall that the wedge product of two 1-forms α and β satisfies $\alpha \wedge \beta(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$.

(d) [4 pts] Relation (4) makes it strikingly easy to prove some rather nontrivial things. For instance, *energy is conserved*: show that for any solution $\mathbf{x}(t)$ of $\dot{\mathbf{x}} = X_H(\mathbf{x})$, $H(\mathbf{x}(t))$ is constant.

(e) [5 pts] Show that $L_{X_H}\omega = 0$, and therefore that the flow $\varphi^t_{X_H}$ is symplectic for all t. By our previous remarks, this implies Liouville's theorem.

The preceding discussion is the beginning of the rather large subject of symplectic geometry, *in which the phase space* $\mathbb{R}^{2n}$ *is replaced by a more general* $2n$*-dimensional manifold* M *with a so-called* symplectic 2-form ω. *A smooth function* $H : M \to \mathbb{R}$ *then defines a Hamiltonian vector field* X_H *via* (4), *and the same argument shows that the flow of* X_H *is symplectic for all* t, *and therefore also volume preserving. A lovely introduction to this subject may be found in V. I. Arnold's* Mathematical Methods of Classical Mechanics.