18.950 SPRING 2007
PROBLEM SET 5
DUE FRIDAY, APRIL 20, 1:00PM

You can hand in the problem set in class or at my office (2-169) anytime before it's due. (Slip it through the slot in the door if I'm not there.)

Reading

Lecture notes §2.3, 2.4.2, 2.4.3, 2.4.6, 2.6–2.8, Appendix B (especially B.1–B.3).

Problems [60 pts total]

1. In this problem we clarify the characterization of the Lie group SO(3) as a group of "rotations" on $\mathbb{R}^3$. Recall that for vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, the *cross product* $\mathbf{v} \times \mathbf{w} \in \mathbb{R}^3$ is defined to be 0 if $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, and is otherwise the unique vector in $\mathbb{R}^3$ with the following properties:

   (i) $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$,
   (ii) $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}| \sin\theta$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$,
   (iii) $(\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w})$ is a positively oriented basis of $\mathbb{R}^3$.

   The operation $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \times \mathbf{w}$ is bilinear, and satisfies the anticommutativity relation $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$ and the *Jacobi identity*

   $$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0.$$

   This makes $\mathbb{R}^3$ with the "bracket" operation $[\mathbf{v}, \mathbf{w}] := \mathbf{v} \times \mathbf{w}$ into a Lie algebra (see Lecture Notes §B.2). Note that the cross product is *not* associative: $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ in general.

   (a) [5 pts] Show that all linear transformations $\mathbf{A} \in \mathrm{SO}(3)$ preserve the cross product, i.e. $\mathbf{A}(\mathbf{v} \times \mathbf{w}) = \mathbf{A}\mathbf{v} \times \mathbf{A}\mathbf{w}$. *Note:* this is not true in general for $\mathbf{A} \in \mathrm{O}(3)$!
   (b) [4 pts] Show that any matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ that preserves lengths also preserves angles, in other words if $|\mathbf{A}\mathbf{v}| = |\mathbf{v}|$ for all $\mathbf{v} \in \mathbb{R}^3$, then $\mathbf{A} \in \mathrm{O}(3)$. *Hint:* consider the dot product of $\mathbf{A}(\mathbf{v} + \mathbf{w})$ with itself.

   Now define a linear map $\Phi : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$, which associates to any $\mathbf{v} \in \mathbb{R}^3$ the matrix of the linear transformation

   $$\Phi(\mathbf{v}) : \mathbb{R}^3 \to \mathbb{R}^3 : \mathbf{w} \mapsto \mathbf{v} \times \mathbf{w}.$$

   (c) [8 pts] Show that for any $\mathbf{u} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, $e^{t\Phi(\mathbf{u})} \in \mathrm{SO}(3)$. *Hint:* for $\mathbf{v}_0 \in \mathbb{R}^3$, the path of vectors $\mathbf{v}(t) = e^{t\Phi(\mathbf{u})}\mathbf{v}_0$ can be characterized as the unique solution to a certain differential equation with initial condition $\mathbf{v}(0) = \mathbf{v}_0$. Use this to show that $\Phi(\mathbf{u})$ preserves lengths and apply part (b).
   (d) [8 pts] Conclude from part (c) that $\Phi(\mathbf{u}) \in \mathfrak{so}(3)$ for all $\mathbf{u} \in \mathbb{R}^3$. In fact, show that Φ is a Lie algebra isomorphism $\mathbb{R}^3 \to \mathfrak{so}(3)$: this means it is a vector space isomorphism and also preserves the bracket operations, so in this case, $\Phi(\mathbf{v} \times \mathbf{w}) = [\Phi(\mathbf{v}), \Phi(\mathbf{w})] = \Phi(\mathbf{v})\Phi(\mathbf{w}) - \Phi(\mathbf{w})\Phi(\mathbf{v})$.
   (e) [5 pts] Let $\mathrm{Aut}(\mathfrak{so}(3))$ denote the group of invertible linear transformations from $\mathfrak{so}(3)$ to itself; this is an open subset of the vector space $\mathrm{End}(\mathfrak{so}(3))$ and thus a Lie group. The *adjoint representation* of SO(3) is the group homomorphism

   $$\mathrm{SO}(3) \to \mathrm{Aut}(\mathfrak{so}(3)) : \mathbf{A} \mapsto \mathrm{Ad}_{\mathbf{A}}$$

defined by $\mathrm{Ad}_{\mathbf{A}}(\mathbf{B}) = \mathbf{A}\mathbf{B}\mathbf{A}^{-1}$. Take a moment to convince yourself that this does define a group homomorphism, i.e. $\mathrm{Ad}_{\mathbf{AB}} = \mathrm{Ad}_{\mathbf{A}} \circ \mathrm{Ad}_{\mathbf{B}}$ (no need to write it down). Then show that for all $\mathbf{A} \in \mathrm{SO}(3)$ and $\mathbf{v} \in \mathbb{R}^3$,

$$\Phi(\mathbf{A}\mathbf{v}) = \mathrm{Ad}_{\mathbf{A}}(\Phi(\mathbf{v})).$$

In other words, the natural action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$ becomes the adjoint representation on $\mathfrak{so}(3)$ under the isomorphism $\Phi : \mathbb{R}^3 \to \mathfrak{so}(3)$.

(f) [5 pts] Show that for any $\mathbf{v} \in \mathbb{R}^3$, $e^{t\Phi(\mathbf{v})}\mathbf{v} = \mathbf{v}$. *Hint:* you might find useful the fact that any matrix $\mathbf{A}$ commutes with $e^{\mathbf{A}}$, though you don't necessarily have to do it this way.

The above leads to the conclusion that for $\mathbf{v} \neq 0$, $e^{t\Phi(\mathbf{v})}$ is a rotation about the axis spanned by $\mathbf{v}$, with rotation angle proportional to t and $|\mathbf{v}|$. Since $\mathfrak{so}(3) \to \mathrm{SO}(3) : \mathbf{A} \mapsto e^{\mathbf{A}}$ is an immersion and $\mathrm{SO}(3)$ is connected, a simple topological argument shows that every transformation in $\mathrm{SO}(3)$ can be written in this way. We also deduce from this a new interpretation of the cross product: in some sense $\mathbf{v} \times \mathbf{w}$ measures the degree to which rotations about $\mathbf{v}$ and rotations about $\mathbf{w}$ fail to commute. In particular, two rotations commute if and only if they rotate about the same axis.

2. [10 pts] Recall that every n-dimensional complex vector space can also be considered a *real* vector space of dimension $2n$. Similarly a complex vector bundle $E \to M$ of rank m is also a real bundle of rank $2m$. Show that every real vector bundle obtained in this way is orientable. You may assume the following fact: *every complex basis of $\mathbb{C}^m$ is continuously deformable into every other complex basis.*

3. Define S^{2n-1} as the unit sphere in $\mathbb{C}^n$ and observe that unitary linear transformations $\mathbf{A} \in \mathrm{U}(n)$ map S^{2n-1} to itself. Then if $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{C}^n$, define the map

$$\pi : \mathrm{U}(n) \to S^{2n-1} : \mathbf{A} \mapsto \mathbf{A}\mathbf{e}_1.$$

(a) [5 pts] Given a matrix $\mathbf{B} \in \mathrm{U}(n-1)$ identify this with the slightly larger matrix

$$\begin{pmatrix} 1 & \\ & \mathbf{B} \end{pmatrix} \in \mathrm{U}(n),$$

and show that the map

$$\mathrm{U}(n) \times \mathrm{U}(n-1) \to \mathrm{U}(n) : (\mathbf{A}, \mathbf{B}) \mapsto \mathbf{AB}$$

defines a right action of $\mathrm{U}(n-1)$ on $\mathrm{U}(n)$ which preserves the level sets $\pi^{-1}(\mathbf{v})$ for $\mathbf{v} \in S^{2n-1}$.

(b) [5 pts] Show that π is surjective and for each $\mathbf{v} \in S^{2n-1}$, $\pi^{-1}(\mathbf{v})$ is a smooth manifold diffeomorphic to $\mathrm{U}(n-1)$.

(c) [5 pts] From the above considerations, it's not hard to believe that $\pi : \mathrm{U}(n) \to S^{2n-1}$ is a principal $\mathrm{U}(n-1)$-bundle: to prove this one must construct appropriate local trivializations. In light of the group action, it suffices in fact to construct local sections near each point, which isn't hard. Let's bypass this detail and ask instead the following question: given that $\pi : \mathrm{U}(n) \to S^{2n-1}$ is a principal $\mathrm{U}(n-1)$-bundle, is it the *frame bundle* of some Hermitian vector bundle of rank $n-1$? The answer is yes—identify the vector bundle in question, and explain.