18.950 SPRING 2007
PROBLEM SET 6
DUE FRIDAY, MAY 4, 1:00PM

You can hand in the problem set in class or at my office (2-169) anytime before it's due. (Slip it through the slot in the door if I'm not there.)

Reading

Lecture notes §3.1–3.3 (3.3.4 optional), 3.4 (skim), Chapter 4.

Problems [50 pts total]

1. Assume M is an n-dimensional manifold. Recall that if $\lambda \in \Omega^1(M)$ is a 1-form, then the bilinear map $\omega : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to C^\infty(M)$,

$$\omega(X,Y) = L_X\left(\lambda(Y)\right) - L_Y\left(\lambda(X)\right) - \lambda([X,Y])$$

is C^∞-linear in both variables and therefore defines a tensor field; in fact it's clearly antisymmetric, thus $\omega \in \Omega^2(M)$. As you may have suspected or read in a book by this point, ω is $d\lambda$. It's time to prove this.

(a) [5 pts] Choosing local coordinates $(x^1, \ldots, x^n)$ and writing $\lambda = \lambda_i \, dx^i$, $d\lambda = (d\lambda)_{ij} \, dx^i \otimes dx^j$, show that

$$(d\lambda)_{ij} = \partial_i \lambda_j - \partial_j \lambda_i.$$

Use this to verify explicitly that $d^2 f = 0$ for any $f \in C^\infty(M)$.

(b) [5 pts] By intelligent choice of vector fields X and Y, show that this coordinate formula implies the coordinate free formula

$$d\lambda(X,Y) = L_X\left(\lambda(Y)\right) - L_Y\left(\lambda(X)\right) - \lambda([X,Y]). \tag{1}$$

This formula has a generalization for forms of higher degree; see p. 213 in Spivak. Admittedly, I personally have never found the general version very useful, though the simple case above has saved my life many times.

Here's a different kind of generalization which will come in useful when we discuss curvature on bundles. For a vector bundle $E \to M$ of rank m, a *bundle-valued differential k-form* $\omega \in \Omega^k(M,E)$ is a smooth, antisymmetric multilinear bundle map

$$\omega : \underbrace{TM \oplus \ldots \oplus TM}_{k} \to E,$$

in other words the same thing as an ordinary differential form, except that for $X_1, \ldots, X_k \in T_pM$, $\omega(X_1, \ldots, X_k)$ is not a real number but rather a vector in the fiber E_p. For instance, given an ordinary differential form $\alpha \in \Omega^k(M)$ and a section $v \in \Gamma(E)$, one can define a bundle-valued form $v\alpha \in \Omega^k(M,E)$ by

$$v\alpha(X_1, \ldots, X_k) = \alpha(X_1, \ldots, X_k) \cdot v(p)$$

for $X_1, \ldots, X_k \in T_pM$. In local coordinates $(x^1, \ldots, x^n)$, any $\omega \in \Omega^k(M,E)$ can be written via components $\omega_{i_1 \ldots i_k}$ as

$$\omega = \omega_{i_1 \ldots i_k} \, dx^{i_1} \otimes \ldots \otimes dx^{i_k} = \sum_{1 \le i_1 < \ldots < i_k \le n} \omega_{i_1 \ldots i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

where the $\omega_{i_1\dots i_k}$ are now no longer real-valued functions but rather sections of E. (Note that they are necessarily antisymmetric under interchange of indices.)

Writing $\Omega^0(M) = C^\infty(M)$, if we think of the differential d as a linear map $\Omega^0(M) \to \Omega^1(M) : f \mapsto df$, then the natural generalization in this setting is to define $\Omega^0(M, E) = \Gamma(E)$ and consider a covariant derivative operator

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E),$$

which assigns to any section $v \in \Gamma(E)$ the bundle-valued 1-form $\nabla v(X) := \nabla_X v$. One can use this to define a *covariant exterior derivative*

$$d_\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$$

which matches ∇ on $\Omega^0(M, E)$ and satisfies

$$d_\nabla \left(v \; dx^{i_1} \wedge \ldots \wedge dx^{i_k} \right) = \nabla v \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \nabla_j v \; dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

for any section v. Observe that all of this depends on a choice of connection for E.

(c) [5 pts] Show that in local coordinates, bundle-valued 1-forms $\lambda \in \Omega^1(M, E)$ satisfy

$$(d_\nabla \lambda)_{ij} = \nabla_i \lambda_j - \nabla_j \lambda_i.$$

(d) [5 pts] Prove the corresponding coordinate free expression

$$d_\nabla \lambda(X, Y) = \nabla_X (\lambda(Y)) - \nabla_Y (\lambda(X)) - \lambda([X, Y]).$$

Note: part of your task here is to prove that the right hand side gives a well defined bundle-valued 2-form; use C^∞-linearity!

(e) [5 pts] Choosing a local frame $(e_{(1)}, \ldots, e_{(m)})$ for E, we can write sections v as $v^i e_{(i)}$ and add an upper index to bundle-valued forms $\omega \in \Omega^k(M, E)$ so that

$$\omega = \omega^i_{j_1 \dots j_k} e_{(i)} \; dx^{j_1} \otimes \ldots \otimes dx^{j_k}.$$

The covariant derivative on $\Gamma(E)$ now takes the form

$$(\nabla_j v)^i = \partial_j v^i + \Gamma^i_{jk} v^k,$$

where the *Christoffel symbols* Γ^i_{jk} are scalar-valued functions with indices $i, k \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Show that for any section v,

$$(d^2_\nabla v)^k_{ij} = \left(\partial_i \Gamma^k_{j\ell} - \partial_j \Gamma^k_{i\ell} + \Gamma^k_{im} \Gamma^m_{j\ell} - \Gamma^k_{jm} \Gamma^m_{i\ell} \right) v^\ell.$$

This expression is not zero, in general, so $d^2_\nabla \neq 0$. Notice however that it doesn't actually depend on any derivatives of v. In fact it tells us much less about the section than about the connection itself: this turns out to be one way of writing the curvature *defined by ∇!*

2. A *symplectic form* on a $2n$-dimensional manifold M is a 2-form ω which is both closed ($dw = 0$) and *nondegenerate*: the latter means that for all $p \in M$, there is no $Y \in T_p M$ such that $\omega(Y, Z) = 0$ for all $Z \in T_p M$. In this situation, for each $p \in M$ the linear map

$$\Phi_\omega : T_p M \to T_p^* M$$

defined by $\Phi_\omega(Y) Z = \omega(Y, Z)$ is injective, and is therefore an isomorphism (both spaces have the same dimension). Thus for any smooth function $H : M \to \mathbb{R}$, there is a unique vector field $X_H \in \mathrm{Vec}(M)$ such that

$$dH = -\omega(X_H, \cdot) = -\iota_{X_H} \omega.$$

We call $H \in C^\infty(M)$ in this context a *Hamiltonian* and X_H the corresponding *Hamiltonian vector field*. An example was seen in the last problem on Problem Set 4, where we had $M = \mathbb{R}^{2n}$ with global coordinates $(q^1, \ldots, q^n, p^1, \ldots, p^n)$ and the "standard" symplectic form

$$\omega_0 := \sum_{j=1}^n dp^j \wedge dq^j.$$

(a) [5 pts] Show that ω_0 is indeed a symplectic form on $\mathbb{R}^{2n}$.

(b) [10 pts] Assume M is any $2n$-dimensional manifold with an *exact* symplectic form $\omega = d\lambda$. Given $t_0 < t_1 \in \mathbb{R}$ and two points $p_0, p_1 \in M$, denote

$$C^\infty([t_0, t_1], M; p_0, p_1) = \{\gamma \in C^\infty([t_0, t_1], M) \mid \gamma(t_0) = p_0 \text{ and } \gamma(t_1) = p_1\}.$$

For a given Hamiltonian $H \in C^\infty(M)$, we define the *symplectic action funcational* by

$$\mathcal{A}_H : C^\infty([t_0, t_1], M; p_0, p_1) \to \mathbb{R} : \gamma \mapsto \int_{[t_0,t_1]} \left[\gamma^*\lambda - H(\gamma(t))\ dt\right].$$

Show that a path $\gamma \in C^\infty([t_0, t_1], M; p_0, p_1)$ is stationary for $\mathcal{A}_H$ if and only if it is an orbit of X_H, i.e. $\dot\gamma(t) = X_H(\gamma(t))$. *Hint:* relation (1) might be useful.

(c) [10 pts] The geodesic equation on an n-manifold M is locally a system of n second order differential equations, but one can use the following trick to turn it into a system of $2n$ first order equations. Observe that the total space of the tangent bundle TM is a $2n$-dimensional manifold. We will use the notation $(q, p) \in TM$, where $q \in M$ and $p \in T_qM$; note that since the value of q constrains p to a particular fiber, we cannot quite think of q and p as independent sets of variables, though with a little caution this point of view can be helpful. The tangent space $T_{(q,p)}(TM)$ can now be understood as follows. Given a Riemannian metric $\langle\ ,\ \rangle$ on M, denote by ∇ the associated Levi-Civita connection. In fiber bundle terms, this defines a horizontal-vertical splitting

$$T_{(q,p)}(TM) = H_{(q,p)}(TM) \oplus V_{(q,p)}(TM).$$

Now the bundle projection $\pi : TM \to M$ defines an isomorphism $\pi_* : H_{(q,p)}(TM) \to T_qM$, and there is already a natural isomorphism of $V_{(q,p)}(TM)$ to the fiber T_qM (the vertical tangent space is after all just a tangent space to the fiber). In this way our connection defines an isomorphism

$$T_{(q,p)}(TM) \cong T_qM \oplus T_qM,$$

which encodes the breaking of a tangent vector to TM into its horizontal and vertical parts respectively. In particular, if $(q(t), p(t)) \in TM$ is a smooth path (so $p(t) \in T_{q(t)}M$ is a vector field along $q(t)$), its velocity vector at time 0 can be understood by taking the horizontal part $\dot q(0) \in TM$ along with the vertical part: the latter is the projection of $\dot p(t) \in T_p(TM)$ to the vertical subspace, in other words the covariant derivative. We have therefore

$$\frac{d}{dt}(q(t), p(t))\Big|_{t=0} = (\dot q(0), \nabla_t p(0)) \in T_{q(0)}M \oplus T_{q(0)}M \cong T_{(q(0),p(0))}(TM).$$

Now using this identification, the Riemannian metric on M defines a 1-form $\lambda \in \Omega^1(TM)$ by

$$\lambda(\xi, \eta) = \langle p, \xi\rangle$$

for $(\xi, \eta) \in T_{(q,p)}(TM)$. It turns out that $d\lambda$ is a symplectic form on TM (prove it in your spare time, if you wish). Define the Hamiltonian function

$$H : TM \to \mathbb{R} : (q, p) \mapsto \frac{1}{2}\langle p, p\rangle.$$

Then for smooth paths $(q(t), p(t)) \in TM$ between fixed end points (q_0, p_0) and (q_1, p_1), the symplectic action functional $\mathcal{A}_H : C^\infty([t_0, t_1], TM; (q_0, p_0), (q_1, p_1)) \to \mathbb{R}$ takes the form

$$\mathcal{A}_H(q, p) = \int_{t_0}^{t_1} \left(\langle p(t), \dot q(t)\rangle - \frac{1}{2}\langle p(t), p(t)\rangle\right)\ dt.$$

Show that $(q(t), p(t))$ is an orbit of X_H if and only if it satisfies the equations

$$\begin{aligned} \dot q(t) &= p(t) \\ \nabla_t p(t) &= 0. \end{aligned}$$

Explain why this is equivalent to the geodesic equation for the path $q(t) \in M$.

Note: there are at least two ways you could go about this. Since you have explicit formulas for λ *and* H*, one way would be to use* $dH = -\iota_{X_H} d\lambda$ *to derive a formula for* X_H*. It might however be easier to take the variational approach, and derive conditions for the explicit form of the action functional above to be stationary. You will need to use the description of tangent spaces to* TM *explained above, along with the symmetry of the connection, compatibility with the metric, and integration by parts.*