18.950 SPRING 2007
TAKEHOME MIDTERM 1
DUE THURSDAY, MARCH 22, 11:00AM

Important note: The due date for this midterm is strict—late submissions will not be accepted. If preferred you can also bring it to my office (2-169) *before* it's due. If you don't come to lecture on the 22nd but I find your midterm in my office afterwards, I will not grade it, but I will laugh at you.

Reading

The following portions of Spivak cover topics on differential forms and integration which we have discussed in lecture, some of which also appear on this midterm:

Chapter 7, skip pp. 215–217 (middle), casually skim pp. 220–226.
Chapter 8, up to p. 263 (middle).

Problems [200 pts total]

1. (a) [10 pts] Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \le 0 \end{cases}$$

is smooth.

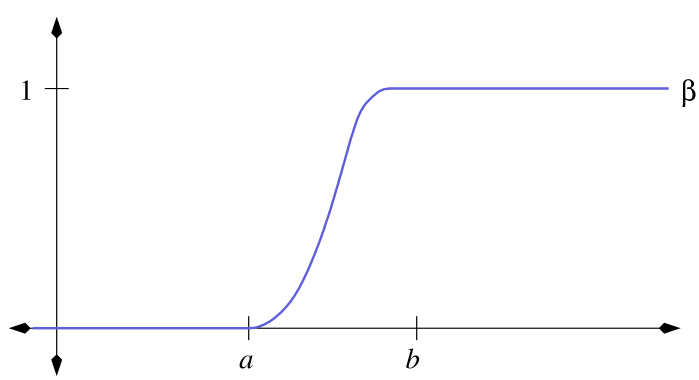
Note that the existence of a smooth function that vanishes on some open subset but not everywhere is by no means obvious. Observe that the Taylor series for $f(x)$ about 0 never equals $f(x)$ for $x > 0$. If you know a little complex analysis, you may be aware that no such thing ever happens with analytic *functions: if $\mathcal{U} \subset \mathbb{C}$ is a nonempty open subset and $f : \mathbb{C} \to \mathbb{C}$ is analytic with $f|_{\mathcal{U}} \equiv 0$, then f is zero everywhere. Morally, this means the set of smooth functions is much larger than the set of analytic functions.*

(b) [5 pts] Using the function f above, find another smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) > 0$ for all $x \in (0, 1)$ but $g(x) = 0$ for $x \le 0$ and $x \ge 1$.

(c) [5 pts] Given g as above, show that $h : \mathbb{R} \to \mathbb{R} : x \mapsto \int_0^x g(t)\, dt$ is also a smooth function. Now for any $a < b \in \mathbb{R}$, use this to find a smooth function $\beta : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $\beta(x) \in [0, 1]$ for all x
- $\beta(x) = 0$ for all $x \le a$
- $\beta(x) = 1$ for all $x \ge b$

A function with these properties is often called a *bump function*.



(d) [5 pts] For any $\mathbf{x}_0 \in \mathbb{R}^n$ and $\epsilon > 0$, find a smooth function $\beta_{\mathbf{x}_0} : \mathbb{R}^n \to \mathbb{R}$ such that $\beta_{\mathbf{x}_0}(\mathbf{x}_0) = 1$ and $\beta_{\mathbf{x}_0}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with $|\mathbf{x} - \mathbf{x}_0| \geq \epsilon$. *Hint:* you might find it useful to note that the function $\mathbb{R}^n \to \mathbb{R} : \mathbf{x} \mapsto |\mathbf{x}|^2$ is smooth.

(e) [10 pts] Let M be a smooth n-manifold and $X \subset M$ a *discrete* subset; this means that every $p \in X$ is contained in some open set $\mathcal{U} \subset M$ such that $\mathcal{U} \cap X = \{p\}$. Then if $g : X \to \mathbb{R}$ is an arbitrary map, construct a smooth function $f : M \to \mathbb{R}$ such that $f(p) = g(p)$ for all $p \in X$. Put another way: *there exist smooth functions having any desired values on any discrete subset.*

2. Given a smooth map $f : M \to N$, a point $q \in N$ is called a *regular value* of f if for every $p \in f^{-1}(q) \subset M$, the derivative $Tf|_{T_pM} : T_pM \to T_qN$ is surjective. Values $q \in N$ that are not regular are called *critical values* of f.

(a) [5 pts] Let $f : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \mapsto x^2 + y^2 - z^2$. Which are the regular values of f, and which values are critical?

(b) [10 pts] Describe (draw) the level sets $f^{-1}(-1)$, $f^{-1}(0)$ and $f^{-1}(1)$. Which of these are submanifolds of $\mathbb{R}^3$? Explain what the answer to this question has to do with the implicit function theorem.

(c) [10 pts] Let $\nabla f \in \mathrm{Vec}(\mathbb{R}^3)$ denote the *gradient vector field*

$$\nabla f(p) = \frac{\partial f}{\partial x}(p)\ \partial_x + \frac{\partial f}{\partial y}(p)\ \partial_y + \frac{\partial f}{\partial z}(p)\ \partial_z,$$

and for all points $p \in \mathbb{R}^3$ with $\nabla f(p) \neq 0$, define

$$X(p) = \frac{\nabla f(p)}{|\nabla f(p)|^2} \in T_p\mathbb{R}^3 = \mathbb{R}^3.$$

Since $X(p)$ is not necessarily well defined everywhere, one must be careful in talking about the flow φ_X^t of X, but ignore this detail for the moment. Assuming it's well defined, what is

$$\frac{d}{dt} f(\varphi_X^t(p))?$$

What does this tell you about the image of a level set $f^{-1}(a)$ under φ_X^t?

(d) [5 pts] Consider any two values $a, b \in \mathbb{R}$ for which $f^{-1}(a)$ and $f^{-1}(b)$ are both manifolds. Under what circumstances are these manifolds diffeomorphic? You needn't prove it, but formulate a conjecture along the following lines:

> *Given a smooth map $f : M \to N$, two level sets $f^{-1}(p)$ and $f^{-1}(q)$ are diffeomorphic submanifolds of M if ...*

In the case $f : \mathbb{R}^3 \to \mathbb{R}$ considered above, how could you use the flow φ_X^t to prove this? (Again, there are technicalities to be dealt with regarding the existence of the flow, but I'm only asking for a heuristic argument.)

Note: the conjecture should not say "if and only if"; that would be asking a bit much.

3. Let M and N be smooth manifolds with a diffeomorphism $\varphi : M \to N$. If $X \in \mathrm{Vec}(M)$, we define the *push-forward* $\varphi_*X \in \mathrm{Vec}(N)$ according to the formula $\varphi_*X(\varphi(p)) = T\varphi(X(p))$, or in other words

$$\varphi_*X(q) = T\varphi(X(\varphi^{-1}(q))).$$

(a) [10 pts] Show that for any $X \in \mathrm{Vec}(M)$, $f \in C^\infty(N)$ and $q \in N$,

$$(L_{\varphi_*X} f)(q) = (L_X(f \circ \varphi))(\varphi^{-1}(q)).$$

(b) [10 pts] Use the result of part (a) to show that for any $X, Y \in \mathrm{Vec}(M)$,

$$[\varphi_*X, \varphi_*Y] = \varphi_*[X, Y].$$

In fancy terms, this shows that $\varphi_* : \mathrm{Vec}(M) \to \mathrm{Vec}(N)$ is a *Lie algebra homomorphism.*

4. As we mentioned once in lecture, there is a natural 1-form on S^1 which we often denote by $d\theta$, even though it is not technically the differential of a smooth function $\theta : S^1 \to \mathbb{R}$. We define it as follows. Regard S^1 as the unit circle in $\mathbb{R}^2$, and for any $p \in S^1$, $X \in T_pS^1$, choose a smooth path $\gamma : (-\epsilon, \epsilon) \to S^1$ with $\gamma(0) = p$, $\dot{\gamma}(0) = X$. Restricting the parameter to a suitably small neighborhood of 0, we can define the angular coordinate $\theta(\gamma(t))$ so that it depends smoothly on $t \in (-\epsilon, \epsilon)$; any alternative choice is related to ours by a constant offset of $2\pi k$ for some $k \in \mathbb{Z}$. Thus the number

$$d\theta(X) := \left.\frac{d}{dt}\theta(\gamma(t))\right|_{t=0}$$

doesn't depend on the choice. The goal of this problem is to compute $\int_{S^1} d\theta$.

This integral is defined via a *partition of unity*, that is, a (possibly infinite or uncountable) collection of open subsets $\mathcal{U}_i \subset S^1$ such that $\bigcup_i \mathcal{U}_i = S^1$, together with smooth functions $\psi_i : S^1 \to [0, 1]$ such that:

- For each $p \in S^1$, only finitely many of the numbers $\psi_i(p)$ are nonzero, and $\sum_i \psi_i(p) = 1$. (Note that the first statement guarantees that the second is well defined.)
- For each $\mathcal{U}_i$, the closure of $\{p \in S^1 \mid \psi_i(p) \neq 0\}$ is contained in $\mathcal{U}_i$. (Recall that the *closure* of a subset A in any metric space X is the set of all points $x \in X$ for which there exists a sequence $x_j \in A$ with $x_j \to x$. It is necessarily a closed subset of X.)

(a) [10 pts] Show that such a collection $\{(\mathcal{U}_i, \psi_i)\}$ exists on S^1. In fact, find such a collection which is *finite* and such that each $\mathcal{U}_i$ is contained in the image of some orientation preserving 1-chain $c_i : [0, 1] \to S^1$. (Assume S^1 is oriented so that the "positive" direction is counterclockwise.) You might find the *bump function* of Problem 1 useful.

(b) [10 pts] Recall that every 1-form on $[0, 1]$ can be written as $f\ dt$ for some smooth function $f : [0, 1] \to \mathbb{R}$, and the integral of this form is defined by

$$\int_{[0,1]} f\ dt := \int_0^1 f(t)\ dt,$$

where the right hand side is the standard Riemann integral of single variable calculus. Then for any 1-form λ on S^1 that is zero outside the image of some orientation preserving 1-chain $c : [0, 1] \to S^1$, one defines

$$\int_{S^1} \lambda := \int_{[0,1]} c^*\lambda.$$

Now, given the choices $\{(\mathcal{U}_i, \psi_i, c_i)\}$ above, we have 1-forms $\psi_i\ d\theta$ on S^1 that are each zero outside of $c_i([0, 1])$, and by construction $\sum_i \psi_i\ d\theta = d\theta$. Thus we define

$$\int_{S^1} d\theta = \sum_i \int_{S^1} \psi_i\ d\theta.$$

Compute this.

5. [5 pts] Use Stokes' theorem to show that the 1-form $d\theta$ on S^1 from Problem 4 is not actually the differential of any smooth function on S^1. (Note: this is stronger than the observation that $\theta : S^1 \to \mathbb{R}$ is not a well defined smooth function—the claim is that *there is no* smooth function $f : S^1 \to \mathbb{R}$ whose differential is $d\theta$). *Hint:* this is easy.

6. Here's a fact that you may or may not already know: given a set of column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$, the volume of the parallelopiped spanned by these vectors is

$$\pm \det \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$$

where the sign is determined according to whether $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ defines a positively or negatively oriented basis of $\mathbb{R}^n$. To see that this formula is correct, recall that the space of antisymmetric k-forms on $\mathbb{R}^n$ has dimension $\frac{n!}{k!(n-k)!}$, which is 1 in particular for $k = n$. Since the determinant itself defines a nonzero n-form, every other volume form on $\mathbb{R}^n$ is a constant multiple of this. We see then that the determinant is *the right* volume form, because choosing the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ (these give a positive basis by definition), one gets the volume of the unit cube:

$$\det \begin{pmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{pmatrix} = \det \mathbb{1} = 1.$$

Here $\mathbb{1}$ denotes the n-by-n identity matrix.

Now denote by $\mathbb{R}^{n\times n}$ the vector space of real n-by-n matrices, and define the subset

$$G = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \det(\mathbf{A}) = \pm 1\}.$$

This set is closed under matrix multiplication and inversion since $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$, thus in algebraic terms, it forms a *group*.

(a) [5 pts] Describe a geometric interpretation for the group G, i.e. what geometric notion involving vectors in $\mathbb{R}^n$ is *preserved* by all linear transformations in G, and only by these transformations?

(b) [5 pts] A somewhat more important group is the *special linear group*

$$\mathrm{SL}(n, \mathbb{R}) = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \det(\mathbf{A}) = 1\}.$$

What must you add to the answer to part (a) to give a geometric interpretation of $\mathrm{SL}(n, \mathbb{R})$?

(c) [10 pts] $\mathrm{SL}(n, \mathbb{R})$ is a smooth submanifold of $\mathbb{R}^{n\times n}$. Use the implicit function theorem to prove this in the case $n = 2$. What would you guess is $\dim \mathrm{SL}(n, \mathbb{R})$ for general n? Prove the answer for $n = 2$.

(d) [15 pts] Suppose $\mathbf{A} : (-\epsilon, \epsilon) \to \mathbb{R}^{n\times n}$ is a smooth path with $\mathbf{A}(0) = \mathbb{1}$. Then show that

$$\left.\frac{d}{dt} \det(\mathbf{A}(t))\right|_{t=0} = \operatorname{tr}(\dot{\mathbf{A}}(0)),$$

where tr denotes the *trace*, i.e. the sum of the diagonal elements. *Hint:* think of $\mathbf{A}(t)$ as an n-tuple of smooth paths of column vectors $\mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)$ with $\mathbf{v}_j(0) = \mathbf{e}_j$. Then $\det(\mathbf{A}(t))$ is an antisymmetric n-form on these vectors and can be written via fixed (i.e. t-independent) components

$$\det(\mathbf{A}(t)) = \det \begin{pmatrix} \mathbf{v}_1(t) & \cdots & \mathbf{v}_n(t) \end{pmatrix} = \omega_{i_1 \ldots i_n} \mathbf{v}_1^{i_1}(t) \ldots \mathbf{v}_n^{i_n}(t).$$

(Alert: implied summations!) Use the product rule to differentiate this with respect to t and rewrite the answer as a sum of determinants, then simplify. If you get stuck, just work out the $n = 2$ case.

(e) [10 pts] Show that $T_{\mathbb{1}} \mathrm{SL}(n, \mathbb{R}) = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \operatorname{tr}(\mathbf{A}) = 0\}$.

(f) [5 pts] Show that if $\mathbf{A}, \mathbf{B} \in T_{\mathbb{1}} \mathrm{SL}(n, \mathbb{R})$, then the *commutator* $[\mathbf{A}, \mathbf{B}] := \mathbf{AB} - \mathbf{BA}$ is also in $T_{\mathbb{1}} \mathrm{SL}(n, \mathbb{R})$.

(g) [5 pts] Recall the orthogonal group $\mathrm{O}(n) = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}^T\mathbf{A} = \mathbb{1}\}$, for which

$$T_{\mathbb{1}} \mathrm{O}(n) = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}^T + \mathbf{A} = 0\}.$$

Show that $T_{\mathbb{1}} \mathrm{O}(n)$ is also preserved by the bracket operation $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$.

This means that $T_{\mathbb{1}} \mathrm{SL}(n, \mathbb{R})$ *and* $T_{\mathbb{1}} \mathrm{O}(n)$ *with the bracket operation* [,] *defined by commutation are* Lie algebras. *Both are examples of a more general phenomenon involving* Lie groups, *that is, groups that are also manifolds: in general the tangent space to a Lie group at the identity has a natural Lie algebra structure. There is an infinite dimensional analog of this statement which we've already seen: if* $\mathrm{Diff}(M)$ *is the group of diffeomorphisms* $M \to M$, *with the "product" operation defined by composition* $\varphi \circ \psi$, *then in principle, the tangent space to* $\mathrm{Diff}(M)$ *at the identity is* $\mathrm{Vec}(M)$, *the space of vector fields. The Lie bracket on vector fields is then the natural Lie algebra structure induced on* $\mathrm{Vec}(M)$ *by the group structure of* $\mathrm{Diff}(M)$.

7. Recall that for $X \in \mathrm{Vec}(\mathbb{R}^n)$, the *divergence* of X is the real valued function

$$\nabla \cdot X = \sum_{i=1}^{n} \partial_i X^i.$$

Unfortunately, this expression turns out not to be invariant under coordinate transformations (if you want to prove this to yourself see Problem (3c) on Problem Set 3 for inspiration). So there is no well defined notion of divergence for vector fields on general n-manifolds M. However, we can define it if M has a little extra structure, namely, a *volume form*. Assume M is oriented, and it comes equipped with a nowhere zero n-form $\mu \in \Omega^n(M)$. Given $X \in \mathrm{Vec}(M)$, we can define a function $\mathrm{div}(X) \in C^\infty(M)$ using the *Lie derivative* of μ with respect to X: this is the n-form

$$L_X \mu = \frac{d}{dt}(\varphi_X^t)^* \mu \Big|_{t=0}$$

where $\varphi_X^t : M \to M$ is the flow of X; in other words for vectors $Y_1, \ldots, Y_n \in T_pM$ at $p \in M$, $L_X\mu$ is defined by

$$L_X\mu(Y_1, \ldots, Y_n) = \frac{d}{dt}\mu(T\varphi_X^t(Y_1), \ldots, T\varphi_X^t(Y_n))\Big|_{t=0}.$$

Now since $L_X\mu$ is an n-form and $\dim \Lambda^n T_p^* M = 1$, there is a unique smooth function $\mathrm{div}(X) : M \to \mathbb{R}$ defined by the condition

$$L_X\mu = \mathrm{div}(X) \cdot \mu.$$

We call $\mathrm{div}(X)$ the *divergence* of X, a term which will be justified below.

(a) [10 pts] Show that the new definition gives $\mathrm{div}(X) = \nabla \cdot X$ if $X \in \mathrm{Vec}(\mathbb{R}^n)$ and we use the standard volume form $\mu = dx^1 \wedge \ldots \wedge dx^n$. (*Hint:* you might find it very useful that partial derivatives commute.)

(b) [10 pts] For any n-dimensional submanifold $\Omega \subset M$ with boundary, define the volume $\mathrm{Vol}(\Omega) := \int_\Omega \mu$. Show that

$$\frac{d}{dt}\mathrm{Vol}(\varphi_X^t(\Omega))\Big|_{t=0} = \int_\Omega \mathrm{div}(X)\ \mu.$$

Hint: remember the change of variables formula $\int_{\psi(\Omega)} \omega = \int_\Omega \psi^*\omega$ whenever ω is an n-form and ψ is an orientation preserving diffeomorphism on to its image $\psi(\Omega) \subset M$. The flow of a vector field is *always* orientation preserving (give a brief argument as to why).

(c) [10 pts] Here is an exceptionally useful formula (proven in Spivak p. 235, Problem 18): for any vector field X and k-form ω,

$$L_X\omega = d\iota_X\omega + \iota_X d\omega,$$

where $\iota_X\omega \in \Omega^{k-1}(M)$ is the so-called *interior product* $\iota_X\omega(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1})$, and $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the *exterior derivative.* Recall now that for the n-form μ, $d\mu = 0$ since all $(n+1)$-forms on an n-manifold are trivial. Use this and Stokes' theorem to prove the generalized *Gauss divergence theorem*:

$$\int_\Omega \mathrm{div}(X)\ \mu = \int_{\partial\Omega} \iota_X\mu.$$

As we saw in lecture, when $M = \mathbb{R}^n$ the right hand side can be interpreted as the surface integral

$$\int_{\partial\Omega} X \cdot \nu\ dA,$$

where ν is the unit outward normal vector to $\partial\Omega$.

(d) [5 pts] A diffeomorphism $\psi : M \to M$ is called *volume preserving* if for every n-dimensional submanifold Ω with boundary,

$$\mathrm{Vol}(\Omega) = \mathrm{Vol}(\psi(\Omega)).$$

Under what conditions on a vector field $X \in \mathrm{Vec}(M)$ is its flow φ_X^t volume preserving for all t?