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TAKEHOME MIDTERM 2
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Reading

Lecture notes, Chapters 5 and 6.

The problems below include topics covered in Chapter 5 but not Chapter 6. The latter will however be fair game for the final.

Problems [150 pts total]

1. We've often discussed flows of vector fields but avoided the question of whether these exist globally. An exception was Problem 3 on Problem Set 2, where we saw an example of a vector field on $\mathbb{R}$ for which most solutions go to infinity in finite time. An even simpler example is the manifold $M = (0, 1)$ with vector field $X(x) = \frac{\partial}{\partial x}$: then the solutions $x(t) = t + c$ run out the edge of M in finite time. One thing both of these examples have in common is that the manifolds are noncompact, and in fact there is a useful general theorem about differential equations which can be stated thus:

   Theorem 1. *For any smooth vector field X on a* compact *manifold M (without boundary), the flow φ_X^t is globally well defined and smooth for all $t \in \mathbb{R}$.*

   One can of course ask the same question about geodesics on a Riemannian manifold; these are a slightly different animal since the differential equation is second order. Clearly there is a danger: e.g. if (M, g) is any proper open subset of $\mathbb{R}^n$ with the standard Euclidean metric, one can easily find a geodesic that runs out of the subset in finite time. We say that a Riemannian manifold is *geodesically complete* if this never happens, i.e. every geodesic $\gamma(t) \in M$ has domain $\mathbb{R}$, rather than just an open subinterval of $\mathbb{R}$. Observing that our example above is once again a noncompact manifold, we'd like to prove the following:

   Theorem 2. *Every compact Riemannian manifold is geodesically complete.*

   One approach to this uses the idea of Problem (2c) on Problem Set 6 to change the geodesic equation on M into the flow of a Hamiltonian vector field X_H on TM. The drawback you may notice immediately is that TM is *never* compact, even if M is; indeed, the tangent spaces T_pM are all noncompact. This is however an easily surmountable obstacle.

   (a) [10 pts] Use Theorem 1 to prove Theorem 2 by viewing geodesics as orbits of a Hamiltonian vector field on TM as in Problem Set 6. *Hint:* you may assume that if $E \to M$ is any fiber bundle with M and the standard fiber both compact, then the total space E is also compact. With this in mind, see Problem Set 4, Problem (4d).

   (b) [5 pts] It is perfectly possible for a noncompact manifold also to be geodesically complete, e.g. this is true for $\mathbb{R}^n$ with the standard Euclidean metric. However, find a Riemannian metric g on $\mathbb{R}^n$ such that $(\mathbb{R}^n, g)$ is *not* geodesically complete. *Hint:* $\mathbb{R}^n$ is diffeomorphic to an open ball.

2. Recall that the Poincaré half plane $(\mathbb{H}, h)$ is the 2-manifold

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

   with Riemannian metric

$$h = \frac{1}{y^2} g_E,$$

where g_E is the standard Euclidean metric on $\mathbb{R}^2$. We showed in lecture that the geodesic equation on $(\mathbb{H}, h)$ takes the form

$$\begin{aligned} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} &= 0 \\ \ddot{y} + \frac{1}{y}\left(\dot{x}^2 - \dot{y}^2\right) &= 0. \end{aligned} \tag{1}$$

(a) [10 pts] Show that for any constants $x_0 \in \mathbb{R}$ and $r > 0$, Equations (1) admit solutions of the form

$$(x(t), y(t)) = (x_0, y(t))$$

for some function $y(t) > 0$, as well as

$$(x(t), y(t)) = (x_0 + r\cos\theta(t), r\sin\theta(t)).$$

for some function $\theta(t) \in (0, \pi)$.

(b) [10 pts] Prove that the solutions of part (a) give *all* geodesics on $(\mathbb{H}, h)$, and that any two points in $\mathbb{H}$ can be joined by a *unique* geodesic. *Note: you can prove this mostly with pictures.*

(c) [10 pts] Compute the length of the geodesic segment joining (x_0, y_0) and (x_0, y_1) for any $0 < y_0 < y_1$. Compute also the length of the geodesic segment joining $(x_0 + r\cos\theta_0, r\sin\theta_0)$ and $(x_0 + r\cos\theta_1, r\sin\theta_1)$ for any $0 < \theta_0 < \theta_1 < \pi$. Use these results to show that $(\mathbb{H}, h)$ is geodesically complete.

3. An *isometry* of a Riemannian manifold (M, g) is a diffeomorphism $\varphi : M \to M$ such that $\varphi^* g = g$. The isometries of (M, g) form a topological group $\mathrm{Isom}(M, g)$. It's structure in a neighborhood of the identity map can be understood by considering smooth 1-parameter families $\varphi_t \in \mathrm{Isom}(M, g)$ with $\varphi_0 = \mathrm{Id}$. In particular, differentiating this with respect to t at $t = 0$ gives a vector field

$$X(p) = \left.\frac{d}{dt}\varphi_t(p)\right|_{t=0},$$

which must satisfy $L_X g \equiv 0$ due to the condition $\varphi_t^* g = g$. A vector field satisfying this condition is called a *Killing vector field*. Intuitively, we think of it as an "infinitessimal isometry".

(a) [10 pts] Show that if ∇ is any symmetric connection on $TM \to M$, $X \in \mathrm{Vec}(M)$, $\lambda \in \Omega^1(M)$ and $Y \in TM$, then $(L_X\lambda)(Y) = (\nabla_X\lambda)(Y) + \lambda(\nabla_Y X)$. *Hint:* construct a smooth map $\alpha(s,t) \in M$ defined for $(s,t) \in \mathbb{R}^2$ near the origin such that $\partial_s\alpha(s,t) = X(\alpha(s,t))$ and $\partial_t\alpha(0,0) = Y$. It will be crucial that the connection is symmetric, so $\nabla_s\partial_t\alpha = \nabla_t\partial_s\alpha$.

(b) [5 pts] Generalize the above result to the formula

$$\begin{aligned} (L_X T)(Y_1, \ldots, Y_k) = (\nabla_X T)(Y_1, \ldots, Y_k) &+ T(\nabla_{Y_1}X, Y_2, \ldots, Y_k) \\ &+ T(Y_1, \nabla_{Y_2}X, \ldots, Y_k) + \ldots + T(Y_1, \ldots, Y_{k-1}, \nabla_{Y_k}X), \end{aligned}$$

valid for any covariant tensor field $T \in \Gamma(T_k^0 M)$.

(c) [10 pts] Applying the formula above with the Levi-Civita connection so that $\nabla g \equiv 0$, we find $L_X g \equiv 0$ if and only if

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

for all $p \in M$ and $Y, Z \in T_pM$. This is called the *Killing equation.*

The bundle metric on $TM \to M$ defines for each $p \in M$ a so-called *musical isomorphism*

$$\begin{aligned} \flat : T_pM \to\, & T_p^*M : Y \mapsto Y^\flat \\ & Y^\flat(Z) := g(Y, Z). \end{aligned}$$

Thus a vector field $X \in \mathrm{Vec}(M)$ gives rise to a 1-form $X^\flat \in \Omega^1(M)$, and this is a one-to-one correspondence. Show that for any $X \in \mathrm{Vec}(M)$ and $Y \in TM$,

$$(\nabla_Y X)^\flat = \nabla_Y (X^\flat).$$

Then show that X satisfies the Killing equation if and only if the tensor field $\nabla X^\flat \in \Gamma(T^0_2 M)$ defined by $\nabla X^\flat(Y, Z) := (\nabla_Y X^\flat)(Z)$ is antisymmetric.

(d) [20 pts] By the above result, solving the Killing equation is equivalent to finding a 1-form $\lambda \in \Omega^1(M)$ such that

$$\nabla\lambda(Y, Z) + \nabla\lambda(Z, Y) = 0. \tag{2}$$

Suppose $\gamma(s) \in M$ is a geodesic through $\gamma(0) = p \in M$. Show that if $\lambda \in \Omega^1(M)$ satisfies Equation (2), then as a section of T^*M along γ, it also satisfies the second order linear differential equation

$$\nabla_s^2 \lambda = \lambda(R(\dot\gamma, \cdot)\dot\gamma), \tag{3}$$

or to be more precise, for any $Y \in T_{\gamma(s)}M$, $(\nabla_s \nabla_s \lambda)(Y) = \lambda(R(\dot\gamma(s), Y)\dot\gamma(s))$. Here $R(X, Y)Z$ denotes the the curvature tensor $R : TM \oplus TM \oplus TM \to TM$ defined by the Levi-Civita connection on $TM \to M$.

Hint: this is tricky, but here are some tips to get you started. If $Y(s) \in T_{\gamma(s)}M$ is a *parallel* vector field along γ, then show that $(\nabla_s^2 \lambda)(Y) = \partial_s^2 (\lambda(Y))$. One can extend $\gamma(s)$ to a smooth map $\alpha(s, t)$ with $\alpha(s, 0) = \gamma(s)$ so that $\partial_t \alpha(s, 0) = Y(s)$. Then in terms of covariant partial derivatives, Equation (2) says

$$(\nabla_s \lambda)(\partial_t \alpha(s, t)) + (\nabla_t \lambda)(\partial_s \alpha(s, t)) = 0.$$

The rest follows from intelligent use of commuting (or non-commuting) partial derivatives, including the symmetry of the connection and the definition of the curvature tensor.

(e) [8 pts] We now appeal to a general fact about second order linear differential equations: if $\mathbf{x}(t) \in \mathbb{R}^n$ satisfies an equation of the form

$$\ddot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$$

for some smooth family of linear maps $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$, then $\mathbf{x}(t)$ is uniquely determined by its initial position $\mathbf{x}(0)$ and velocity $\dot{\mathbf{x}}(0)$. Use this and Equation (3) to show that if λ satisfies (2) and there is a point $p \in M$ at which $\lambda_p = 0$ and $\nabla\lambda_p = 0$, then $\lambda \equiv 0$.

(f) [7 pts] The previous conclusion together with the linearity of the Killing equation imply a uniqueness statement for the Killing equation: in particular, there is an upper bound (in terms of $\dim M = n$) on the possible dimension of the space of Killing vector fields. What is this bound?

Caution: this is a uniqueness result but says nothing about existence—there are cases where the Killing equation has no nontrivial solutions. The trouble is that while the theory of ODEs guarantees local existence of 1*-forms λ that satisfy Equation* (3) *along a geodesic γ, these need not generally extend to* 1*-forms on an open set that satisfy* (2).

(g) [5 pts] Let us apply the uniqueness result to the case $M = \mathbb{R}^n$ with the standard Euclidean metric $\langle\ ,\ \rangle$ on $T_p\mathbb{R}^n \cong \mathbb{R}^n$. In this case there is a well known family of isometries called the *Euclidean group* $\mathrm{E}(n)$, which consists of all diffeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ of the form

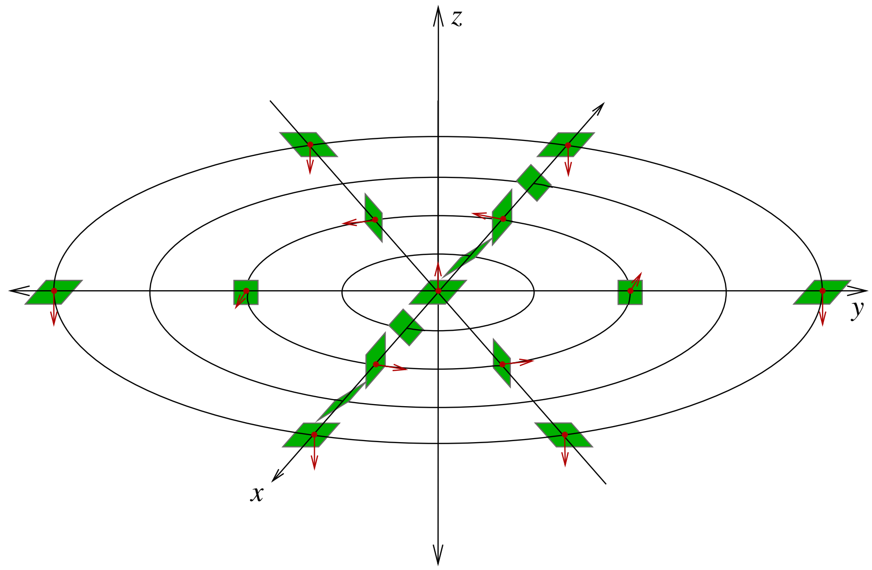
$$\varphi(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$$

for $\mathbf{A} \in \mathrm{O}(n)$ and $\mathbf{b} \in \mathbb{R}^n$. Differentiating any smooth 1-parameter family $\varphi_t \in \mathrm{E}(n)$ with $\varphi_0 = \mathrm{Id}$ gives a Killing vector field

$$X(\mathbf{x}) = \left.\frac{d}{dt}\varphi_t(\mathbf{x})\right|_{t=0}.$$

Show that *all* Killing vector fields on Euclidean n-space are of this form.

Note: one can go slightly further and prove that the Euclidean group contains all isometries of Euclidean space—thus $\mathbb{R}^n$ does not admit any nonlinear *isometries.*



4. Suppose M is an oriented 3-manifold and $\lambda \in \Omega^1(M)$ is nowhere zero, i.e. for all $p \in M$ there exist vectors $X \in T_pM$ with $\lambda(X) \neq 0$. Then at every $p \in M$, the *kernel* $\ker \lambda_p = \{X \in T_pM \mid \lambda(X) = 0\}$ is a 2-dimensional subspace of T_pM, and the union of these for all p defines a smooth 2-dimensional distribution

$$\xi := \ker \lambda \subset TM.$$

(a) [20 pts] Show that the following conditions are equivalent:

i. $\lambda \wedge d\lambda \equiv 0$

ii. For all $p \in M$ there exists $X \in \xi_p$ such that $d\lambda(X, Y) = 0$ for all $Y \in \xi_p$, i.e. the restriction $d\lambda|_\xi$ is *degenerate*.

iii. ξ is integrable.

The 1-form λ is called a *contact form* if $\lambda \wedge d\lambda$ is a volume form; the distribution ξ (called the *contact structure*) is then "as non-integrable as possible." An example on $\mathbb{R}^3$ is shown in the figure above. Such examples can be constructed by the following trick: choosing cylindrical polar coordinates (ρ, ϕ, z) on $\mathbb{R}^3$, choose smooth real-valued functions $f(\rho), g(\rho)$ and define λ at (ρ, ϕ, z) by

$$\lambda = f(\rho)\ dz + g(\rho)\ d\phi. \tag{4}$$

(b) [10 pts] Since the coordinates (ρ, ϕ, z) are not well defined at $\rho = 0$, there is of course some danger that the 1-form defined in Equation (4) might be singular at the z-axis. Show that λ is in fact smooth on all of $\mathbb{R}^3$ and satisfies $\lambda \wedge d\lambda \neq 0$ near the z-axis if we assume $f(\rho) = 1$ and $g(\rho) = \rho^2$ for ρ sufficiently close to 0. *Hint:* convert to Cartesian coordinates.

(c) [10 pts] Assuming f and g take the form desribed above for ρ near 0, show that λ is a contact form if and only if

$$f(\rho)g'(\rho) - f'(\rho)g(\rho) \neq 0$$

for all $\rho > 0$. What does this mean geometrically about the curve $\rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2$? Interpret this in terms of "twisting" of the planes ξ_p as $p \in \mathbb{R}^3$ moves along radial paths away from the z-axis.