CONTACT 3-MANIFOLDS, HOLOMORPHIC CURVES AND INTERSECTION THEORY

EXERCISE SHEET 2

August 28, 2013

(1) Recall that $H_2(\mathbb{C}P^2)$ is generated by an embedded sphere $\mathbb{C}P^1 \subset \mathbb{C}P^2$ with $[\mathbb{C}P^1]\cdot[\mathbb{C}P^1] = 1$. A holomorphic curve $u : \Sigma \to \mathbb{C}P^2$ is said to have **degree** $d \in \mathbb{N}$ if

$$[u] = d[\mathbb{C}P^1].$$

Show that all holomorphic spheres of degree 1 are embedded, and any other simple holomorphic sphere is embedded if and only if it has degree 2.

(2) Suppose Σ and Σ' are compact oriented surfaces with boundary, M is a closed 4-manifold, and $u_s : \Sigma \to M$ and $v_s : \Sigma' \to M$ for $s \in [0,1]$ are smooth homotopies such that for all s,

$$u_s(\partial\Sigma) \cap v_s(\Sigma') = u_s(\Sigma) \cap v_s(\partial\Sigma') = \emptyset.$$

Show that if $u_s \pitchfork v_s$ for $s = 0, 1$, then

$$\sum_{u_0(z)=v_0(\zeta)} \iota(u_0, z; v_0, \zeta) = \sum_{u_1(z)=v_1(\zeta)} \iota(u_1, z; v_1, \zeta),$$

where we denote by $\iota(u, z; v, \zeta) = \pm 1$ the sign of a transverse intersection $u(z) = v(\zeta)$.

(3) Given a compact surface Σ with boundary, a complex line bundle $L \to \Sigma$, and a trivialisation τ of $L|_{\partial\Sigma}$, the **relative first Chern number**

$$c_1^\tau(L) \in \mathbb{Z}$$

can be defined as the signed count of zeroes of a generic section $\eta : \Sigma \to L$ such that $\eta|_{\partial\Sigma}$ is nonzero and constant with respect to τ.

(a) Prove that $c_1^\tau(L)$ as described above does not depend on the choice of the section η.

(b) Prove that the relative first Chern number admits a unique and well-defined extension to higher rank complex vector bundles such that

$$(E, \tau) \cong (E', \tau') \quad \Rightarrow \quad c_1^\tau(E) = c_1^{\tau'}(E')$$

and

$$c_1^{\tau_1 \oplus \tau_2}(E_1 \oplus E_2) = c_1^{\tau_1}(E_1) + c_1^{\tau_2}(E_2).$$

(4) Suppose (W, ω) is a symplectic cobordism with convex boundary $(M_+, \xi_+ = \ker \alpha_+)$ and concave boundary $(M_-, \xi_- = \ker \alpha_-)$, $(\widehat{W}, \hat{\omega})$ is its completion, and J is an almost complex structure on $\widehat{W}$ that is compatible with ω on W, and on the cylindrical ends is translation-invariant and satisfies

$$J(\partial_s) = R_{\alpha_\pm}, \qquad J(\xi_\pm) = \xi_\pm \quad \text{and } J|_{\xi_\pm} \text{ is compatible with } d\alpha_\pm|_{\xi_\pm},$$

where $R_{\alpha_\pm}$ denotes the Reeb vector field on $M_\pm$ determined by $\alpha_\pm$. For any smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi' > 0$ and $\varphi(s) = s$ near $s = 0$, consider the smooth 2-form

$$\omega_\varphi := \begin{cases} \omega & \text{on } W, \\ d\left(e^{\varphi(s)}\alpha_+\right) & \text{on } [0, \infty) \times M_+, \\ d\left(e^{\varphi(s)}\alpha_-\right) & \text{on } (-\infty, 0] \times M_-. \end{cases}$$

Show that ω_φ is symplectic and J is ω_φ-compatible.