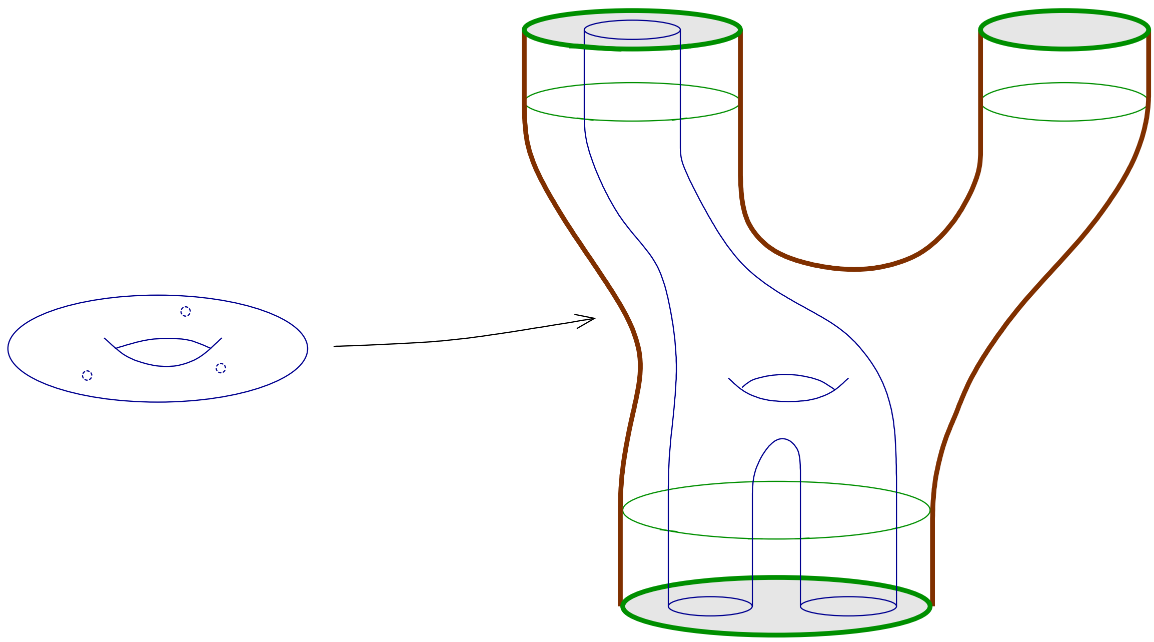
Contact 3-Manifolds, Holomorphic Curves and Intersection Theory

(Durham University, August 2013)



Chris Wendl

University College London

These slides plus detailed lecture notes (in progress) available at:

http://www.homepages.ucl.ac.uk/~ucahcwe/Durham

Background material for Lecture 1

(M^{2n}, ω) is a symplectic manifold:

$$\omega \in \Omega^2(M), \qquad d\omega = 0 \text{ and } \omega^n > 0.$$

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3. Is there a symplectomorphism

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and the space of such symplectic forms is connected.

$(M,\omega) \xrightarrow{\pi} \Sigma$ is then a symplectic fibration.

If $F \cong S^2$, (M,ω) is called a **symplectic ruled surface**.

A more general example

$M \xrightarrow{\pi} \Sigma$ is a **Lefschetz fibration** if it has finitely many critical points $M^{\mathrm{crit}} \subset M$ of the form

$$\pi(z_1, z_2) = z_1^2 + z_2^2$$

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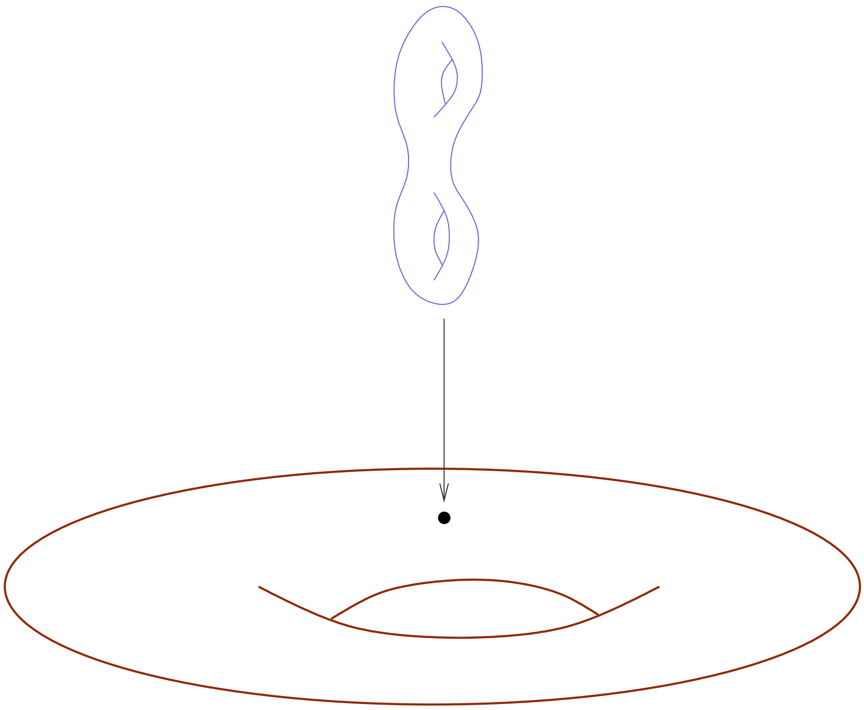

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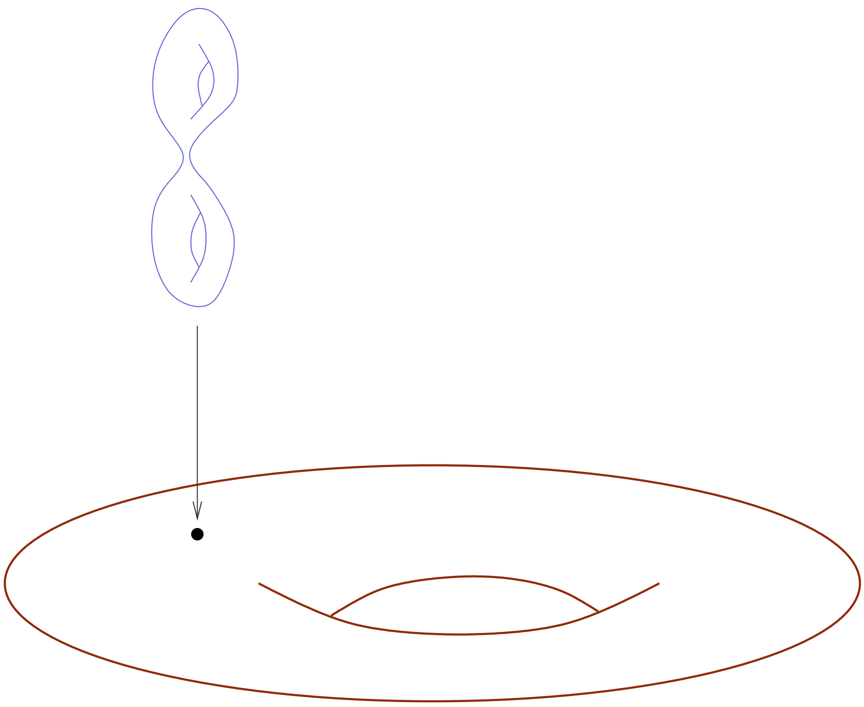

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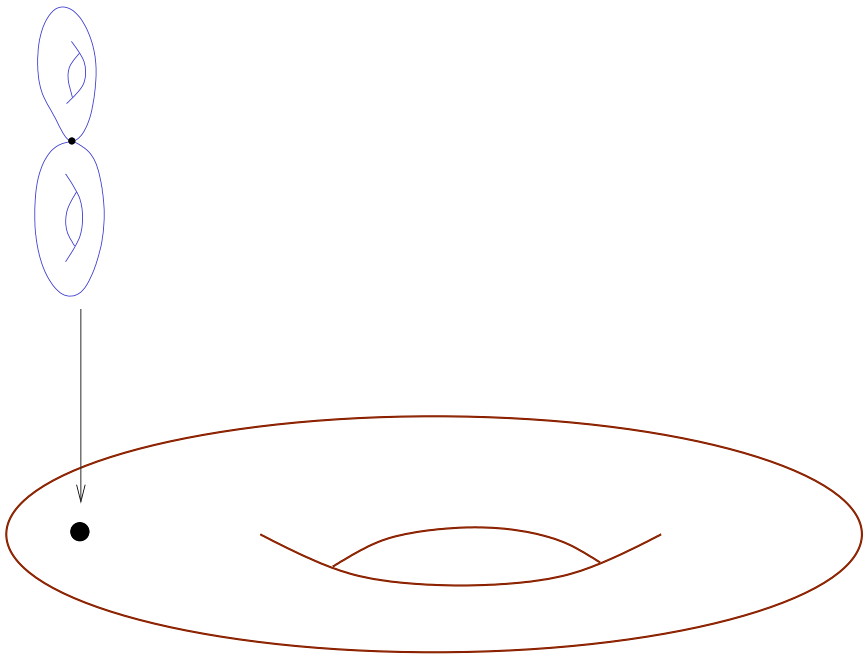

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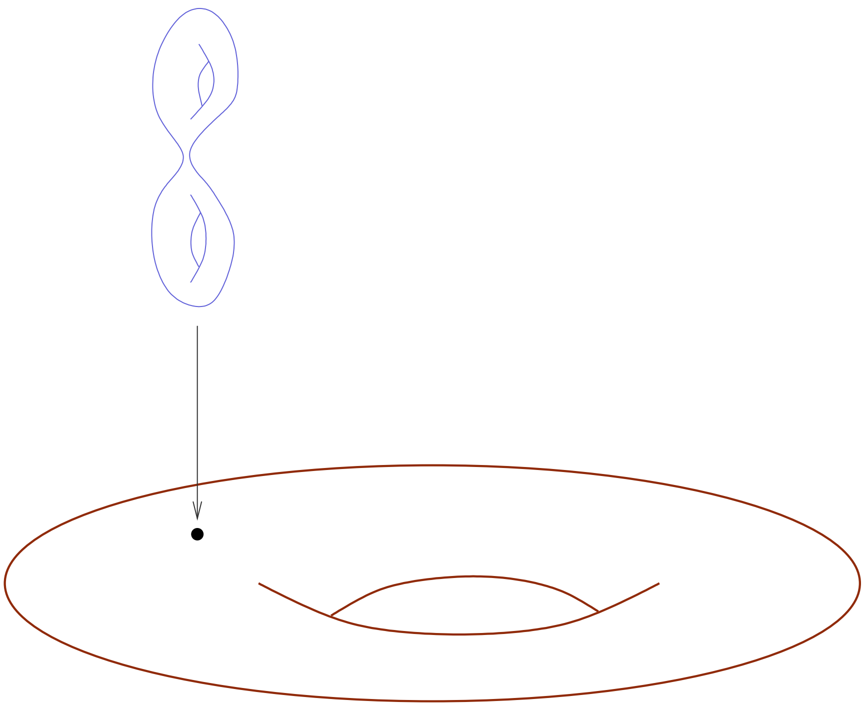

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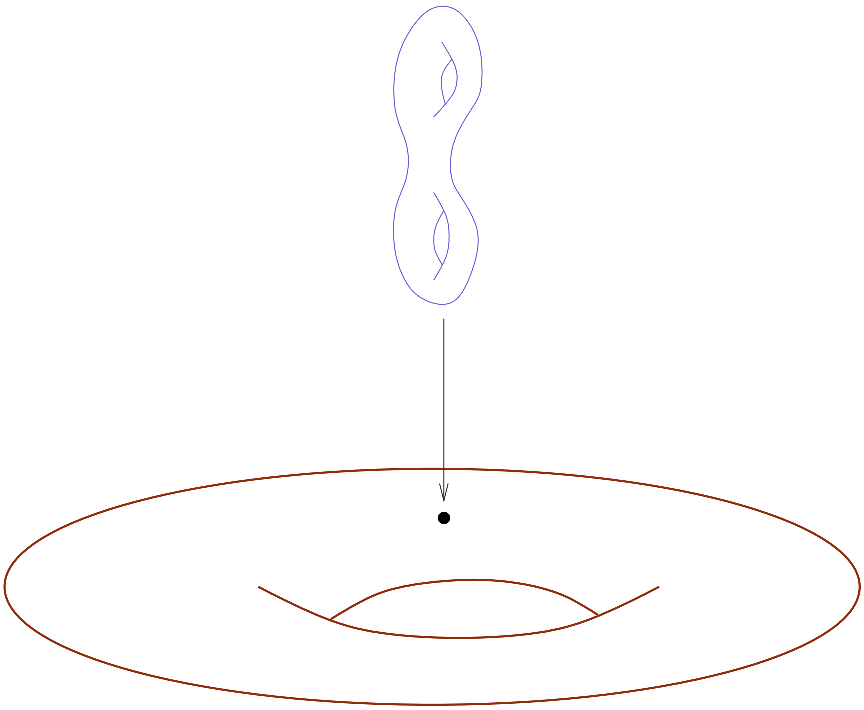

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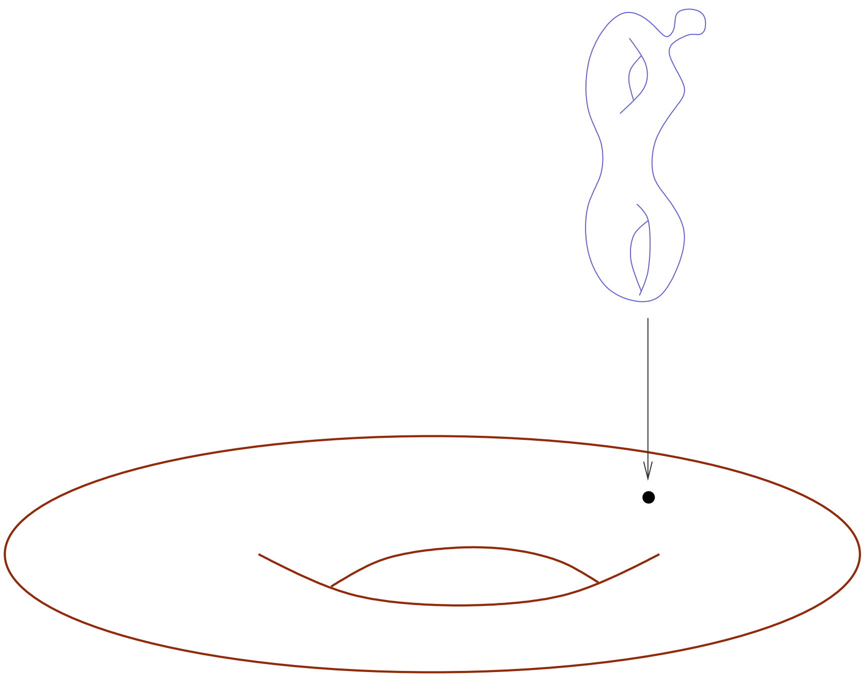

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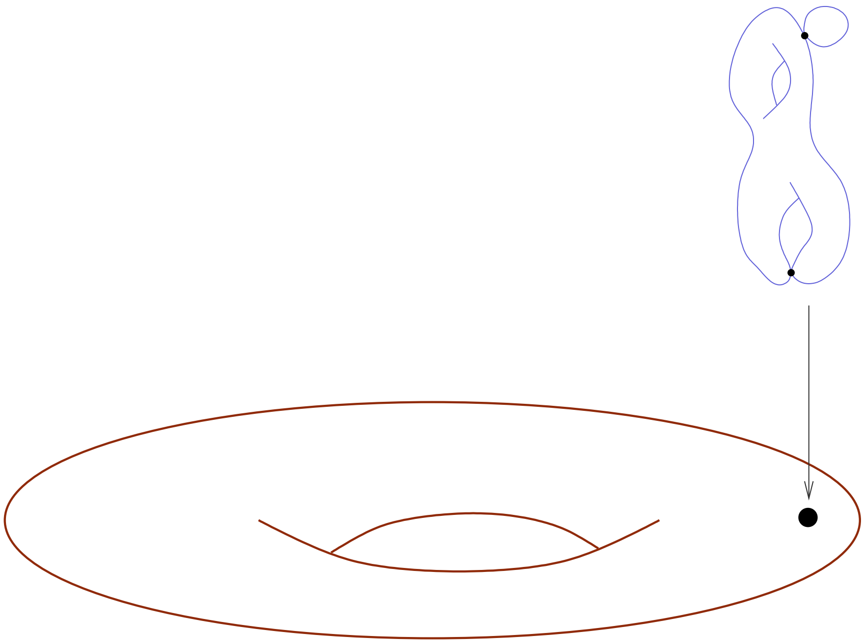

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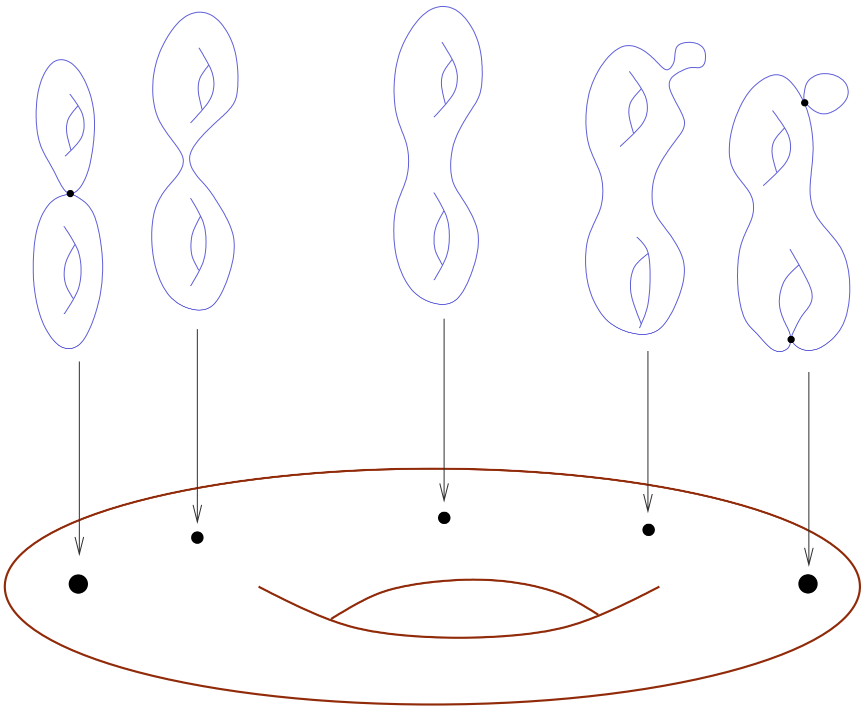

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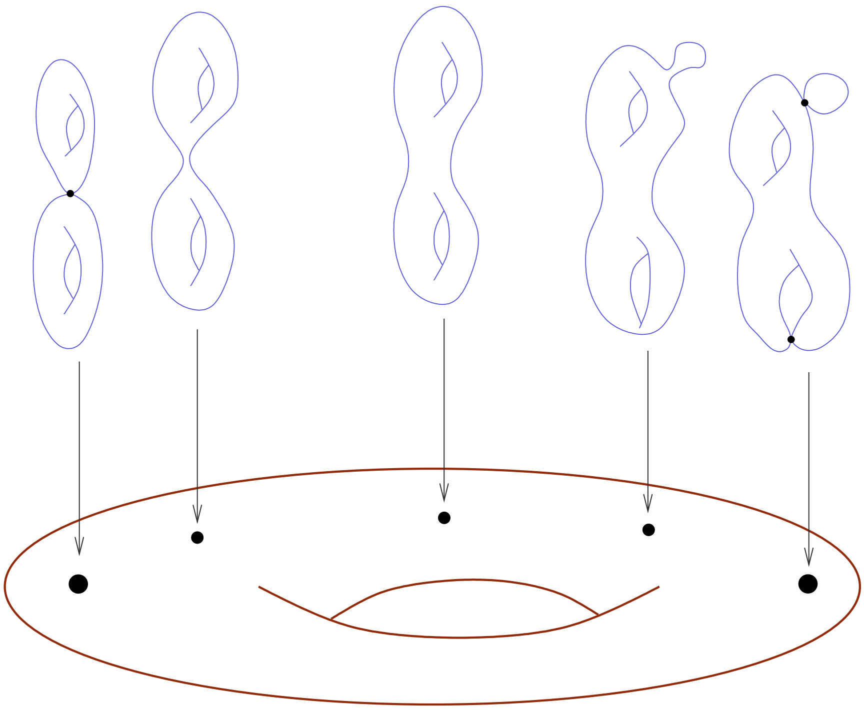

Theorem (Gompf)
Thurston's theorem generalises to Lefschetz fibrations.

We call $(M, \omega) \stackrel{\pi}{\to} \Sigma$ a **symplectic Lefschetz fibration**.

Blowing up

$L \to \mathbb{C}P^1$ tautological line bundle:

$$L_{[z_1:z_2]} := \mathbb{C}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \subset \mathbb{C}^2$$

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For $p \in M^4$ with neighbourhood $\mathcal{N}(p) \subset M$,

$$\begin{aligned} \widehat{M} &:= (M \setminus \mathcal{N}(p)) \cup \mathcal{N}(\mathbb{C}P^1) \\ &\cong M \# \overline{\mathbb{C}P}^2 \end{aligned}$$

This replaces p with an **exceptional sphere**

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If (M, ω) is symplectic, then the **symplectic blowup** $(\widehat{M}, \widehat{\omega})$ is canonical up to symplectic deformation, and the exceptional sphere $E \subset (\widehat{M}, \widehat{\omega})$ is a symplectic submanifold.

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Definition
(M, ω) is **minimal** if it contains no symplectic exceptional spheres
($\Leftrightarrow$ it is not a symplectic blowup).

(slightly off topic but nice to know)

Theorem (McDuff)
Any closed symplectic 4-manifold (M, ω) with a maximal collection of pairwise disjoint exceptional spheres $E_1, \ldots, E_N \subset (M, \omega)$ becomes minimal after "blowing down" along $E_1, \ldots, E_N$.

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Corollary
(M, ω) is symplectomorphic to (a blowup of) a ruled surface.

The tools we will need

An almost complex structure $J : TM \to TM$ ($J^2 = -\mathbb{1}$) is **compatible with ω** if

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For $A \in H_2(M)$ and $g \geq 0$, define the **moduli space**

$$\mathcal{M}_g^A(M, J) := \{(\Sigma, j, u)\} \big/ \text{parametrization},$$

where (Σ, j) is a Riemann surface of genus g, $u : (\Sigma, j) \to (M, J)$ is J-holomorphic, and

$$[u] := u_*[\Sigma] = A.$$

Properties of J-curves in dimension $2n$

(1) Every $u \in \mathcal{M}_g^A(M, J)$ is either **simple** or **multiply covered**

$$u = v \circ \varphi, \quad \varphi : \Sigma \to \Sigma', \quad v : \Sigma' \to M$$

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(2) For generic J, the open subset

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is a manifold with dimension equal to its **virtual dimension**

$$\text{vir-dim}\, \mathcal{M}_g^A(M, J) := (n-3)(2-2g) + 2c_1(A),$$

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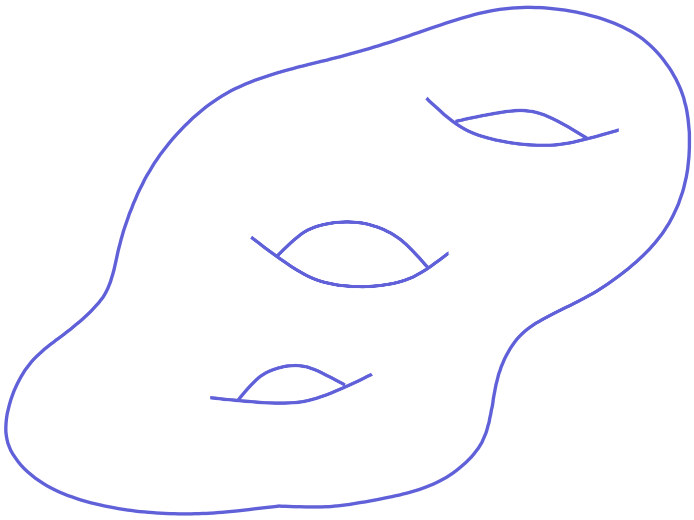
Corollary: J generic, u simple $\Rightarrow$ $\text{ind}(u) \geq 0$.

(3) $\mathcal{M}_g^A(M,J)$ is not compact, but it has a natural **compactification**

$$\overline{\mathcal{M}}_g^A(M,J) := \Big\{ \text{“nodal” curves of arithmetic genus } g, \text{ homologous to } A \Big\}$$



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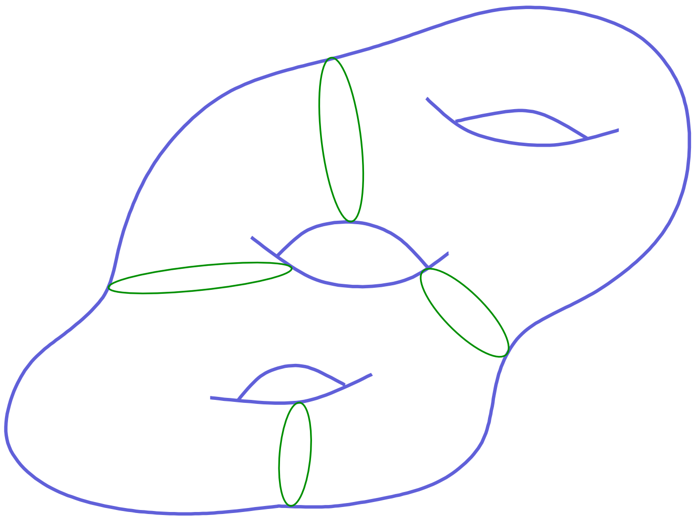

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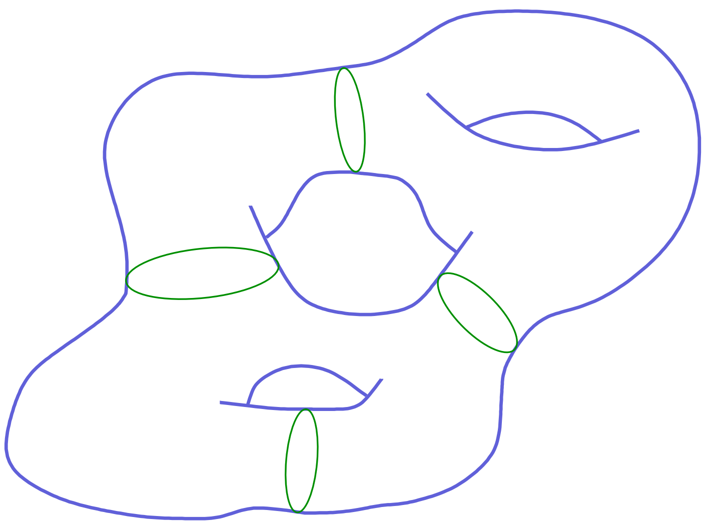

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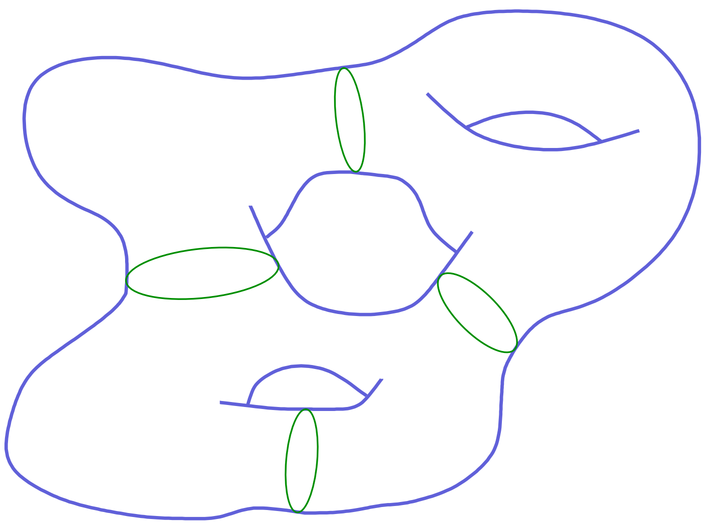

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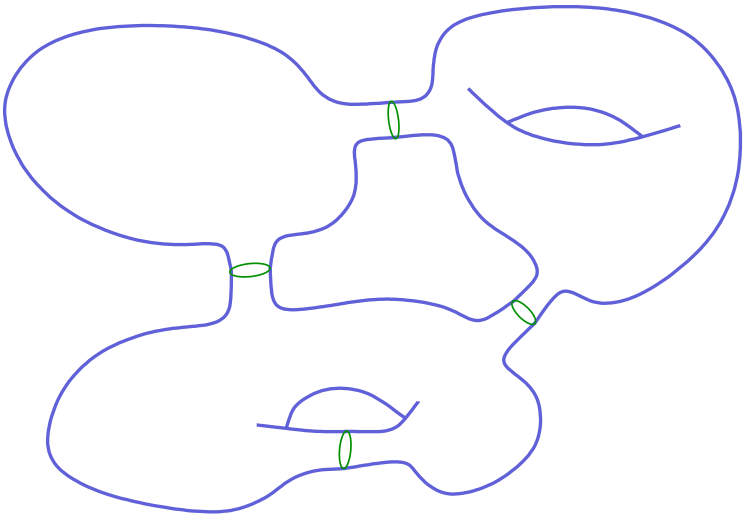

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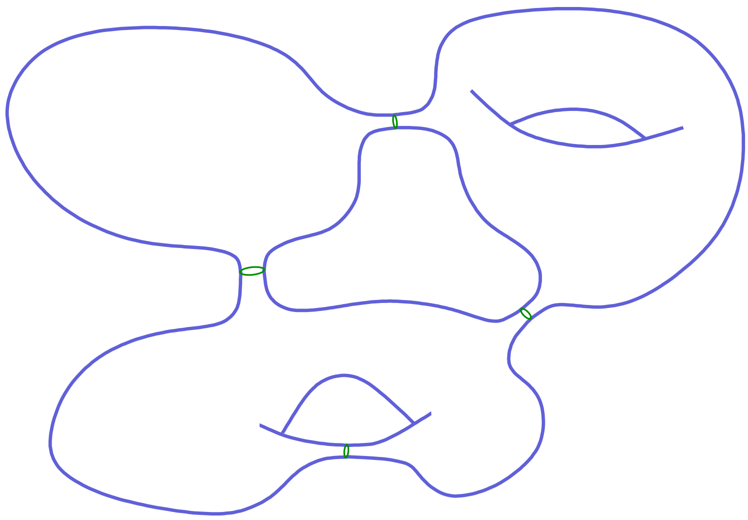

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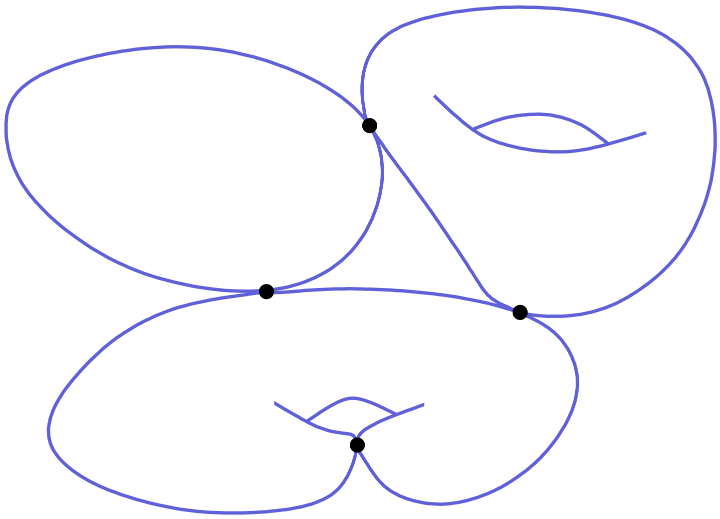

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Proof of McDuff's theorem (sketch)

Inclusion $u_S : S \hookrightarrow (M^4, \omega)$ is symplectic $\Rightarrow$

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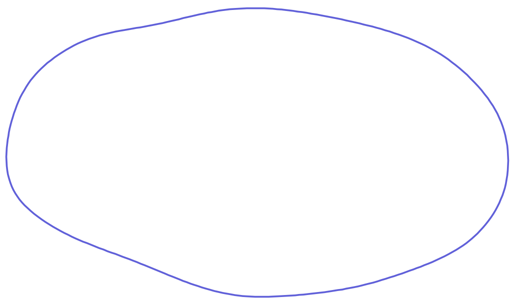
Thus

$$\text{vir-dim}\, \mathcal{M}_0^{[S]}(M, J) = -2 + 2c_1([S]) = 2,$$

$\Rightarrow$ the *simple curves in* $\mathcal{M}_0^{[S]}(M, J)$ *form a smooth 2-parameter family*.

Lemma 1 *(standard* $2n$*-dimensional stuff)* There exists a finite set $\mathscr{B}$ of simple curves $v \in \mathcal{M}_0(M, J)$ with $c_1([v]) > 0$ such that any noncompact sequence $u_k \in \mathcal{M}_0^{[S]}(M, J)$ has a subsequence convergent to a nodal curve with exactly two components $v_+, v_- \in \mathscr{B}$.



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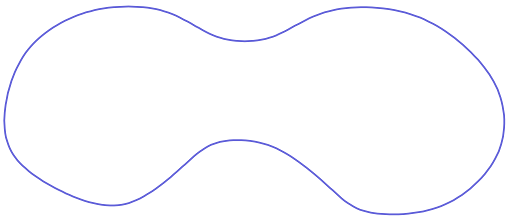

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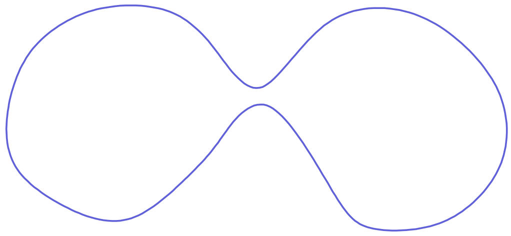

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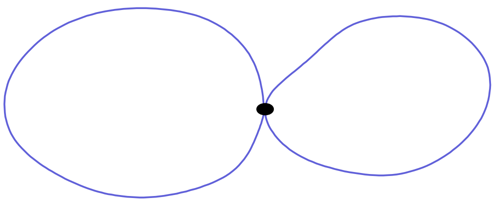

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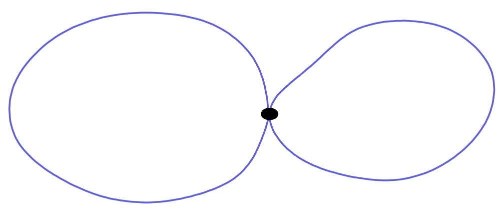

Lemma 2 *(unique to dimension four!)*
For the nodal curves $\{v_+, v_-\}$ in Lemma 1, v_+ and v_- are each embedded, satisfy

$$[v_\pm] \cdot [v_\pm] = -1,$$

and intersect each other exactly once, transversely.
Moreover, all curves in $\mathcal{M}_0^{[S]}(M, J)$ are embedded and disjoint from the nodal curves, and they **foliate an open subset of M**.

Conclusion of the proof

Lemmas 1 and 2 imply that the set

$$\left\{ p \in M \mid p \in \mathrm{im}(u) \text{ for some } u \in \overline{\mathcal{M}}_0^{[S]}(M,J) \right\},$$

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Singular fibres = nodal curves =
two transversely intersecting exceptional spheres disjoint from S

$\Rightarrow$ all fibres are regular if $(M \setminus S, \omega)$ is minimal.