Solution to Take-home Midterm Problem 2(a)

We need to show that if X is compact and Hausdorff, then for any continuous map $\gamma : S^1 = \partial\mathbb{D}^2 \to X$, the space

$$X' = X \cup_\gamma \mathbb{D}^2$$

is also compact and Hausdorff. By definition, $X \cup_\gamma \mathbb{D}^2$ is the quotient $(X \sqcup \mathbb{D}^2)/\sim$ where $z \sim \gamma(z)$ for each $z \in \partial\mathbb{D}^2$, so it carries the quotient topology, meaning that a set $\mathcal{U} \subset X'$ is open if and only if $\pi^{-1}(\mathcal{U}) \subset X \sqcup \mathbb{D}^2$ is open, where π is the quotient projection

$$\pi : X \sqcup \mathbb{D}^2 \to (X \sqcup \mathbb{D}^2)/\sim = X'.$$

Here $X \sqcup \mathbb{D}^2$ carries the disjoint union topology, so every open set in $X \sqcup \mathbb{D}^2$ is simply the union of an open set in X with an open set in $\mathbb{D}^2$.

Compactness is easy. First, since X and $\mathbb{D}^2$ are both compact, $X \sqcup \mathbb{D}^2$ is also compact: indeed, any open cover of $X \sqcup \mathbb{D}^2$ is of the form $\{\mathcal{U}_\alpha \cup \mathcal{V}_\alpha\}_{\alpha \in I}$, where $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is an open cover of X and $\{\mathcal{V}_\alpha\}_{\alpha \in I}$ is an open cover of $\mathbb{D}^2$. Both have finite subcovers, so there exist finite subsets $I_1, I_2 \subset I$ such that $\{\mathcal{U}_\alpha\}_{\alpha \in I_1}$ is still an open cover of X and $\{\mathcal{V}_\alpha\}_{\alpha \in I_2}$ is still an open cover of $\mathbb{D}^2$, and then $\{\mathcal{U}_\alpha \cup \mathcal{V}_\alpha\}_{\alpha \in I_1 \cup I_2}$ is a finite subcover for $X \sqcup \mathbb{D}^2$. (In the same manner, any disjoint union of a finite collection of compact spaces is compact—note that this does not work for infinite disjoint unions.) Now X' is compact since it is the image under π of a compact space, and continuous maps on compact spaces always have compact images.

For the Hausdorff property, we'll make use of the following pair of lemmas:

Lemma 1: $\mathbb{D}^2$ satisfies axiom T_4, i.e. is normal.

Lemma 2: For any open subset $\mathcal{U} \subset X$ and $x \in \mathcal{U}$, there exists an open neighborhood of x whose closure is contained in $\mathcal{U}$.

The first lemma is true because $\mathbb{D}^2$ is a metric space, see Problem Set 4 #5(c). The second follows from Problem Set 3 #6 since X is Hausdorff and compact, and therefore also locally compact.

We need to show that any two distinct points in X' have disjoint neighborhoods. We'll denote points in X' as equivalence classes $[x]$ where x belongs to either X or $\mathbb{D}^2$. Note that if $[x] \neq [y] \in X'$, then necessarily $x \neq y$. There are three cases two consider.

Case 1: Suppose x and y are distinct points both in the interior $\mathring{\mathbb{D}}^2$ of $\mathbb{D}^2$. Then it suffices to take any pair of disjoint open neighborhoods $x \in \mathcal{U}_x \subset \mathring{\mathbb{D}}^2$ and $y \in \mathcal{U}_y \subset \mathring{\mathbb{D}}^2$, as both project to open subsets in X', i.e. since $\pi^{-1}(\pi(\mathcal{U}_x)) = \mathcal{U}_x$ is open in $X \sqcup \mathbb{D}^2$, $\pi(\mathcal{U}_x)$ is open in X', and similarly for $\pi(\mathcal{U}_y)$. The two sets are then disjoint open neighborhoods of $[x]$ and $[y]$ respectively since $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$ and no point in $\mathring{\mathbb{D}}^2$ is equivalent to any other point.

Case 2: Suppose $x \in \mathring{\mathbb{D}}^2$ and $y \in X$. Since $\mathbb{D}^2$ is normal, we can fix a pair of disjoint open subsets $\mathcal{U}_x, \mathcal{U}_\partial \subset \mathbb{D}^2$ such that $x \in \mathcal{U}_x$ and $\partial\mathbb{D}^2 \subset \mathcal{U}_\partial$. Now choose any open neighborhood $\mathcal{V}_y \subset X$ of y and observe that since γ is continuous, $\gamma^{-1}(\mathcal{V}_y)$ is an open subset of S^1. Note that this does not mean $\gamma^{-1}(\mathcal{V}_y)$ is open in $\mathbb{D}^2$—that would be impossible since $\gamma^{-1}(\mathcal{V}_y)$ is contained in $\partial\mathbb{D}^2$—rather it is open in the subspace topology on $\partial\mathbb{D}^2$ induced by the latter's inclusion in $\mathbb{D}^2$. Concretely, this means there exists an open subset $\mathcal{V}'_y \subset \mathbb{D}^2$ such that $\mathcal{V}'_y \cap \partial\mathbb{D}^2 = \gamma^{-1}(\mathcal{V}_y)$. Then $\mathcal{V}'_y \cap \mathcal{U}_\partial$ is another open subset of $\mathbb{D}^2$ whose intersection with the boundary is $\gamma^{-1}(\mathcal{V}_y)$, but it is also disjoint from $\mathcal{U}_x$. We are therefore led to consider the subsets

$$\pi(\mathcal{U}_x) \quad \text{and} \quad \pi(\mathcal{V}_y \cup (\mathcal{V}'_y \cap \mathcal{U}_\partial)) \subset X',$$

which contain $[x]$ and $[y]$ respectively and are disjoint by construction. The first is open because $\mathcal{U}_x$ is in the interior of $\mathbb{D}^2$, so $\pi^{-1}(\pi(\mathcal{U}_x)) = \mathcal{U}_x$ is open in $X \sqcup \mathbb{D}^2$. The second is also open since $\mathcal{V}_y \cup (\mathcal{V}'_y \cap \mathcal{U}_\partial)$ is open in $X \sqcup \mathbb{D}^2$ and

$$\pi^{-1}(\pi(\mathcal{V}_y \cup (\mathcal{V}'_y \cap \mathcal{U}_\partial)) = \mathcal{V}_y \cup (\mathcal{V}'_y \cap \mathcal{U}_\partial),$$

due to the fact that $(\mathcal{V}'_y \cap \mathcal{U}_\partial) \cap \partial\mathbb{D}^2 = \gamma^{-1}(\mathcal{V}_y)$.

Case 3: Suppose x and y are distinct points in X, and since X is Hausdorff, choose a pair of disjoint open neighborhoods $\mathcal{U}_x, \mathcal{U}_y \subset X$ of x and y respectively. Now using Lemma 2, choose smaller neighborhoods

$\mathcal{V}_x, \mathcal{V}_y$ of x and y whose closures are contained in $\mathcal{U}_x$ and $\mathcal{U}_y$ respectively; this means in particular that $\overline{\mathcal{V}}_x$ and $\overline{\mathcal{V}}_y$ are disjoint. Now $\gamma^{-1}(\mathcal{V}_x)$ and $\gamma^{-1}(\mathcal{V}_y)$ are open subsets of S^1 that similarly have disjoint closures, and using the fact that $\mathbb{D}^2$ is normal, we can find a pair of disjoint open subsets $\mathcal{W}_x, \mathcal{W}_y \subset \mathbb{D}^2$ with

$$\overline{\gamma^{-1}(\mathcal{V}_x)} \subset \mathcal{W}_x \quad \text{and} \quad \overline{\gamma^{-1}(\mathcal{V}_y)} \subset \mathcal{W}_y.$$

Using the definition of the subspace topology again as in Case 2, the openness of $\gamma^{-1}(\mathcal{V}_x)$ and $\gamma^{-1}(\mathcal{V}_y)$ in $\partial\mathbb{D}^2$ means that we can find open subsets $\mathcal{V}'_x, \mathcal{V}'_y \subset \mathbb{D}^2$ such that $\mathcal{V}'_x \cap \partial\mathbb{D}^2 = \gamma^{-1}(\mathcal{V}_x)$ and $\mathcal{V}'_y \cap \partial\mathbb{D}^2 = \gamma^{-1}(\mathcal{V}_y)$. Then $\mathcal{V}'_x \cap \mathcal{W}_x$ and $\mathcal{V}'_y \cap \mathcal{W}_y$ are also open subsets of $\mathbb{D}^2$, which are disjoint and satisfy

$$(\mathcal{V}'_x \cap \mathcal{W}_x) \cap \partial\mathbb{D}^2 = \gamma^{-1}(\mathcal{V}_x) \quad \text{and} \quad (\mathcal{V}'_y \cap \mathcal{W}_y) \cap \partial\mathbb{D}^2 = \gamma^{-1}(\mathcal{V}_y).$$

It follows now as in Case 2 that $\pi(\mathcal{V}_x \cup (\mathcal{V}'_x \cap \mathcal{W}_x))$ and $\pi(\mathcal{V}_y \cup (\mathcal{V}'_y \cap \mathcal{W}_y))$ are disjoint open subsets of X' that contain $[x]$ and $[y]$ respectively.