PROBLEM SET 10
Due: 12.07.2017

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Wednesday lecture.

1. This problem is to make sure you are comfortable with the basic algebraic notions involving abelian groups. Since the most popular examples are $\mathbb{Z}^n$ and $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$, with addition as the group operation, it is conventional to denote the group operation in an arbitrary abelian group G by "+", with the identity element written as "$0 \in G$" and the inverse of an element $g \in G$ denoted by "$-g$". We can then abbreviate $g - h := g + (-h)$ for $g, h \in G$ and think of G as a $\mathbb{Z}$-module, due to the natural action of the ring $\mathbb{Z}$ on G defined by

$$ng := \underbrace{g + \ldots + g}_{n} \text{ for } n > 0, \qquad ng := \underbrace{-g - \ldots - g}_{-n} \text{ for } n < 0, \qquad 0g := 0.$$

If G is the trivial group, we shall indicate this by writing "$G = 0$".

We say that $g \in G$ is a **torsion** element of G if $mg = 0$ for some $m \in \mathbb{N}$, so for instance, every element of $\mathbb{Z}_n$ for $n \in \mathbb{N}$ is torsion, while 0 is the only torsion element in $\mathbb{Z}^n$. We say that G "has torsion" if it contains a torsion element other than 0; otherwise we say G is **torsion free**.

(a) Show that for any abelian group G, the set of torsion elements of G defines a subgroup $G_{\text{tor}} \subset G$, called its **torsion subgroup**, and the quotient G/G_{tor} is torsion free.[1]

An abelian group G is **finitely generated** if it contains a finite subset $S \subset G$ such that every element of G is a sum of elements of S and their inverses, i.e. G is the smallest subgroup of G that contains S. A **basis** of G is a subset $B \subset G$ such that every $g \in G$ can be written as

$$g = \sum_{b \in B} n_b b$$

for a unique set of coefficients $n_b \in \mathbb{Z}$, at most finitely many of which are nonzero. An abelian group is called **free** if it admits a basis.[2] The canonical example is $\mathbb{Z}^n$, for which the standard basis vectors in $\mathbb{R}^n$ form a basis.

(b) $(*)$ Show that if G has torsion, then it is not free.
Hint: The trouble is uniqueness.

(c) Show that for an abelian group G, the following conditions are equivalent: (i) G is finitely generated and free; (ii) G admits a finite basis; (iii) G is isomorphic to $\mathbb{Z}^n$ for some integer $n \geqslant 0$.

(d) Show that an abelian group G is finite if and only if it is finitely generated and $G_{\text{tor}} = G$.

For any collection of abelian groups $\{G_\alpha\}_{\alpha \in I}$, their **direct product** is the abelian group $\bigtimes_{\alpha \in I} G_\alpha$ whose elements $\{g_\alpha\}_{\alpha \in I}$ are functions assigning to each $\alpha \in I$ an element $g_\alpha \in G_\alpha$, with the group operation defined by the obvious formula $\{g_\alpha\}_{\alpha \in I} + \{h_\alpha\}_{\alpha \in I} := \{g_\alpha + h_\alpha\}_{\alpha \in I}$. The **direct sum**

$$\bigoplus_{\alpha \in I} G_\alpha \subset \bigtimes_{\alpha \in I} G_\alpha$$

[1] Note that since G is abelian, every subgroup $H \subset G$ is normal, hence quotient groups G/H are always well defined.

[2] Note that with the exception of $\mathbb{Z}$, a "free abelian group" is not a "free group" in the sense that we've previously discussed, e.g. if S is a set with more than one element, then the *free group on S* (denoted by $F(S)$) is not abelian.

is the subgroup consisting of elements $\{g_\alpha\}_{\alpha\in I}$ such that $g_\alpha = 0$ for all but finitely many $\alpha \in I$. This distinction makes no difference if I is finite, so for instance $G \oplus H$ and $G \times H$ are exactly the same thing, namely the group of ordered pairs with operation $(g, h) + (g', h') := (g + g', h + h')$.

Here are two facts that are non-obvious but hopefully plausible, and we will assume them henceforth:[3]

- Every subgroup of a finitely-generated abelian group is finitely generated.
- Every torsion-free finitely-generated abelian group is free.

(e) Show that every finitely-generated abelian group is isomorphic to $\mathbb{Z}^n \oplus T$ for some integer $n \geqslant 0$ and a finite group $T \cong G_{\mathrm{tor}}$. *Hint: The above results imply that* $G/G_{\mathrm{tor}} \cong \mathbb{Z}^n$ *for some* n.

Given a set S, the **free abelian group on** S is defined as a direct sum of copies of $\mathbb{Z}$, one for each element of S:

$$F^{\mathrm{ab}}(S) := \bigoplus_{s\in S} \mathbb{Z}.$$

Denote by $\langle s\rangle \in F^{\mathrm{ab}}(S)$ the generator $1 \in \mathbb{Z}$ in the copy of $\mathbb{Z}$ corresponding to the element $s \in S$. These elements form a basis of $F^{\mathrm{ab}}(S)$.

(f) Show that for any abelian group H, set S, and map $f : S \to H$, there exists a unique homomorphism $\Phi : F^{\mathrm{ab}}(S) \to H$ such that $\Phi(\langle s\rangle) = f(s)$ for each of the generators $s \in S$.

(g) Show that there is a natural isomorphism between $F^{\mathrm{ab}}(S)$ and the abelianization of the free (non-abelian) group $F(S)$.

Given abelian groups G, H, K, a map $\Phi : G \oplus H \to K$ is called **bilinear** if for every fixed $g_0 \in G$ and $h_0 \in H$, the maps $G \to K : g \mapsto \Phi(g, h_0)$ and $H \to K : h \mapsto \Phi(g_0, h)$ are both homomorphisms.

The **tensor product** of two abelian groups G and H can be defined as the abelian group

$$G \otimes H := F^{\mathrm{ab}}(G \times H)/N$$

where $N \subset F^{\mathrm{ab}}(G \times H)$ is the smallest subgroup containing all elements of the form $\langle(g + g', h)\rangle - \langle(g, h)\rangle - \langle(g', h)\rangle$ and $\langle(g, h + h')\rangle - \langle(g, h)\rangle - \langle(g, h')\rangle$ for $g, g' \in G$ and $h, h' \in H$. We denote the equivalence class represented by $\langle(g, h)\rangle \in F^{\mathrm{ab}}(G \times H)$ in the quotient by

$$g \otimes h \in G \otimes H.$$

(h) Show that the map $G \oplus H \to G \otimes H : (g, h) \mapsto g \otimes h$ is bilinear, and deduce from this that for any $g \in G$ and $h \in H$, $0 \otimes h = g \otimes 0 = 0 \in G \otimes H$.

(i) $(*)$ Show that for any bilinear map $\Phi : G \oplus H \to K$ of abelian groups, there exists a unique homomorphism $\Psi : G \otimes H \to K$ such that $\Phi(g, h) = \Psi(g \otimes h)$ for all $(g, h) \in G \oplus H$.

(j) Show that for any abelian group G, the map $G \to G \otimes \mathbb{Z} : g \mapsto g \otimes 1$ is a group isomorphism. Write down its inverse.
Hint: Part (i) tells you that in order to specify a homomorphism $G \otimes H \to K$*, it suffices to write down a bilinear map* $G \oplus H \to K$.

(k) Find a natural isomorphism from $(G \oplus H) \otimes K$ to $(G \otimes K) \oplus (H \otimes K)$.

(l) Given two sets S and T, find a natural isomorphism from $F^{\mathrm{ab}}(S) \otimes F^{\mathrm{ab}}(T)$ to $F^{\mathrm{ab}}(S \times T)$.

(m) Let $\mathbb{K}$ be a field, regarded as an abelian group with respect to its addition operation. Show that the abelian group $G \otimes \mathbb{K}$ naturally admits the structure of a vector space over $\mathbb{K}$ such that scalar multiplication takes the form

$$\lambda(g \otimes k) = g \otimes (\lambda k)$$

for every $\lambda, k \in \mathbb{K}$ and $g \in G$, and every group homomorphism $\Phi : G \to H$ determines a unique $\mathbb{K}$-linear map $\Psi : G \otimes \mathbb{K} \to H \otimes \mathbb{K}$ such that $\Psi(g \otimes k) = \Phi(g) \otimes k$ for $g \in G$, $k \in \mathbb{K}$.

[3]For proofs, see §I.8 of S. Lang, *Algebra*, revised third edition, Springer GTM, 2002.

(n) Show that for any free abelian group G with basis $B \subset G$ and any field $\mathbb{K}$, the elements $\{b \otimes 1 \mid b \in B\}$ form a basis of the vector space $G \otimes \mathbb{K}$.

(o) $(*)$ Show that if $\mathbb{K}$ is a field of characteristic zero[4] and G is an abelian group in which every element is torsion, then $G \otimes \mathbb{K} = 0$. Show also that this is not true in the case $\mathbb{K} := \mathbb{Z}_2$ (which does not have characteristic zero).

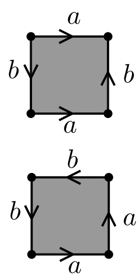
(p) Show that if G is isomorphic to $\mathbb{Z}^n \oplus T$ for some integer $n \geqslant 0$ and finite group T, then for any field $\mathbb{K}$ with characteristic zero, the $\mathbb{K}$-vector space $G \otimes \mathbb{K}$ is isomorphic to $\mathbb{K}^n$. Deduce that G cannot also be isomorphic to $\mathbb{Z}^m \oplus T'$ for a finite group T' and integer $m \neq n$.

The main conclusion of this problem is that the following definition makes sense: the ***rank***

$$\operatorname{rank} G \in \{0, 1, 2, \dots, \infty\}$$

of a finitely-generated abelian group G is the unique integer $n \geqslant 0$ such that $G \cong \mathbb{Z}^n \oplus T$ for a finite group $T \cong G_{\mathrm{tor}}$. Equivalently, $\operatorname{rank} G$ is the number of elements in any basis of the free abelian group G/G_{tor}, or the largest possible number of elements in a linearly independent[5] subset of G, or the dimension over $\mathbb{K}$ of the vector space $G \otimes \mathbb{K}$ for any field $\mathbb{K}$ with characteristic zero. If G/G_{tor} is not finitely generated, we say $\operatorname{rank} G = \infty$.

2. The picture at the right shows two spaces that we've previously seen are both homeomorphic to the Klein bottle (see Problem Set 7 #4). Each also defines a cell complex $X = X^0 \cup X^1 \cup X^2$ consisting of one 0-cell, two 1-cells (labeled a and b) and one 2-cell.



   (a) $(*)$ Write down the chain complexes and compute the cellular homology groups $H_k(X)$ for each of the two cell complexes and $k = 0, 1, 2$. Write each $H_k(X)$ in the form $\mathbb{Z}^n \oplus T$ for $n \geqslant 0$ and a finite group T.

   (b) $(*)$ Do it again with $\mathbb{Z}_2$-coefficients, i.e. compute $H_k(X; \mathbb{Z}_2)$.

   (c) Verify in both cases that $\sum_k (-1)^k \operatorname{rank} H_k(X) = \sum_k (-1)^k \dim_{\mathbb{Z}_2} H_k(X; \mathbb{Z}_2) = 0$.

   (Congratulations, you've just computed the ***Euler characteristic*** *of the Klein bottle!)*

[4] $\mathbb{K}$ has **characteristic zero** if for all $n \in \mathbb{N}$, the n-fold sum $1 + \ldots + 1$ is not 0. The standard examples are $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

[5] A finite subset $S \subset G$ in an abelian group is called **linearly independent** if the only choice of coefficients $n_s \in \mathbb{Z}$ satisfying $\sum_{s \in S} n_s s = 0$ is $n_s = 0$ for all s.