PROBLEM SET 4
Due: 17.05.2017

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Wednesday lecture.

1. In lecture we defined S^1 as the unit circle in $\mathbb{R}^2$ with the subspace topology (induced by the Euclidean metric on $\mathbb{R}^2$). Show that the following spaces with their natural quotient topologies are both homeomorphic to S^1:
   (a) $\mathbb{R}/\mathbb{Z}$, meaning the set of equivalence classes of real numbers where $x \sim y$ means $x - y \in \mathbb{Z}$.
   (b) $(*)$ $[0,1]/\sim$, where $0 \sim 1$.
2. Prove that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic for any $n \geq 2$.
   Hint: If $\mathbb{R}$ and $\mathbb{R}^n$ are homeomorphic, then so are $\mathbb{R} \setminus \{t\}$ and $\mathbb{R}^n \setminus \{x\}$ for some $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Show that one of those spaces is connected and the other is not.
3. Let X be an infinite set, equipped with the cofinite topology.
   (a) Show that X is connected and locally connected.
   (b) Show that if $X = \mathbb{R}$, then X is path-connected and locally path-connected.
   (c) Show that if X is countable, then X is not path-connected.
   Hint: There is a famous theorem of Sierpiński stating that a compact connected Hausdorff space cannot be decomposed as a union of a countable infinity of pairwise disjoint nonempty closed subsets. You'll find various proofs of this on the internet; the special case of a closed interval is somewhat simpler, though still not especially obvious.
4. (a) Show that a space X is connected if and only if every continuous function $f : X \to \{0,1\}$ is constant.
   (b) $(*)$ Prove that if X and Y are both connected, then so is $X \times Y$.[1]
   Hint: Start by showing that for any $x \in X$ and $y \in Y$, the subsets $\{x\} \times Y$ and $X \times \{y\}$ in $X \times Y$ are connected. Then use the criterion in part (a).
   (c) Show that for any (perhaps infinite) collection of path-connected spaces $\{X_\alpha\}_{\alpha \in I}$, the space $\prod_{\alpha \in I} X_\alpha$ is path-connected in the usual product topology.
   Hint: You might find Problem Set 2 #3(d) helpful.
   (d) Consider $\mathbb{R}^{\mathbb{N}}$ with the "box topology" which we discussed in Problem Set 2 #5. Show that the set of all elements $f \in \mathbb{R}^{\mathbb{N}}$ represented as functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy $\lim_{n\to\infty} f(n) = 0$ is both open and closed, hence $\mathbb{R}^{\mathbb{N}}$ in the box topology is not connected (and therefore also not path-connected).
5. (a) Show that a finite topological space satisfies the axiom T_1 if and only if it carries the discrete topology.
   (b) Show that X is a T_2 space (i.e. Hausdorff) if and only if the *diagonal* $\Delta := \{(x,x) \in X \times X\}$ is a closed subset of $X \times X$.
   (c) $(*)$ Show that every metrizable space satisfies the axiom T_4 (i.e. it is *normal*).
   Hint: Given disjoint closed sets $A, A' \subset X$, each $x \in A$ admits a radius $\epsilon_x > 0$ such that the ball $B_{\epsilon_x}(x)$ is disjoint from A', and similarly for points in A' (why?). The unions of all these balls won't quite produce the disjoint neighborhoods you want, but try cutting their radii in half.

[1] The analogous statement about infinite products is also true, but it takes more work to prove it.

6. Suppose X is a Hausdorff space and $\sim$ is an equivalence relation on X. Let $X/\sim$ denote the quotient space equipped with the quotient topology and denote by $\pi : X \to X/\sim$ the canonical projection.

   (a) ($*$) A map $s : X/\sim \to X$ is called a *section* of π if $\pi \circ s$ is the identity map on $X/\sim$. Show that if a continuous section exists, then $X/\sim$ is Hausdorff.

   (b) Let $A \subset X$ be a closed subset and suppose that the equivalence relation is given by $x \sim y$ iff $x = y$ or $x, y \in A$. Show that if X additionally satisfies axiom T_3, then $X/\sim$ is Hausdorff.

   (c) Find an example where X is Hausdorff but $X/\sim$ is not. (Then just for fun, try to construct a continuous section, and notice that you cannot do it.)

7. Prove that $\mathbb{R}^n$ is a simply connected space for every n.

8. (a) ($*$) Given two pointed spaces (X, x) and (Y, y), prove that $\pi_1(X \times Y, (x, y))$ is isomorphic to the product group $\pi_1(X, x) \times \pi_1(Y, y)$.
   Hint: Show that for any paths $\alpha : [0, 1] \to X$ and $\beta : [0, 1] \to Y$ with $\alpha(0) = \alpha(1) = x$ and $\beta(0) = \beta(1) = y$, the path $(\alpha, \beta) : [0, 1] \to X \times Y$ is homotopic with fixed endpoints to a product path $(\alpha, e_y) \cdot (e_x, \beta)$, where e_x and e_y denote the constant paths at x and y respectively.

   (b) Generalize part (a) to the case of an infinite product of pointed spaces (with the product topology).