PROBLEM SET 5
Due: 24.05.2017

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Wednesday lecture.

Special note: *Throughout this problem set, you may use without proof the fact that $\pi_1(S^1) \cong \mathbb{Z}$. A more precise version of this statement is reviewed in #1.*

Problems

1. For a point $z \in \mathbb{C}$ and a continuous map $\gamma : [0,1] \to \mathbb{C} \setminus \{z\}$ with $\gamma(0) = \gamma(1)$, one defines the *winding number* of γ about z as
$$\operatorname{wind}(\gamma; z) = \theta(1) - \theta(0) \in \mathbb{Z}$$
where $\theta : [0,1] \to \mathbb{R}$ is any choice of continuous function such that
$$\gamma(t) = z + r(t)e^{2\pi i\theta(t)}$$
for some function $r : [0,1] \to (0,\infty)$. Notice that since $\gamma(t) \neq z$ for all t, the function $r(t)$ is uniquely determined, and requiring $\theta(t)$ to be continuous makes it unique up to the addition of a constant integer, hence $\theta(1) - \theta(0)$ depends only on the path γ on not on any additional choices. One of the fundamental facts about winding numbers is their important role in the computation of $\pi_1(S^1)$: as we will prove in a few weeks, viewing S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$, the map
$$\pi_1(S^1, 1) \to \mathbb{Z} : [\gamma] \mapsto \operatorname{wind}(\gamma; 0)$$
is an isomorphism to the abelian group $(\mathbb{Z}, +)$. For now, take this fact as given. Assume in the following that $\Omega \subset \mathbb{C}$ is an open set and $f : \Omega \to \mathbb{C}$ is a continuous function.

   (a) Suppose $f(z) = w$ and $w \notin f(\mathcal{U} \setminus \{z\})$ for some neighborhood $\mathcal{U} \subset \Omega$ of z. This implies that the loop $f \circ \gamma_\epsilon$ for $\gamma_\epsilon : [0,1] \to \Omega : t \mapsto z + \epsilon e^{2\pi i t}$ has image in $\mathbb{C} \setminus \{w\}$ for all $\epsilon > 0$ sufficiently small, hence $\operatorname{wind}(f \circ \gamma_\epsilon; w)$ is well defined. Show that for some $\epsilon_0 > 0$, $\operatorname{wind}(f \circ \gamma_\epsilon; w)$ does not depend on ϵ as long as $0 < \epsilon \leq \epsilon_0$.

   (b) $(*)$ Show that if the ball $B_r(z_0)$ of radius $r > 0$ about $z_0 \in \Omega$ has its closure contained in Ω, and the loop $\gamma(t) = z_0 + re^{2\pi i t}$ satisfies $\operatorname{wind}(f \circ \gamma; w) \neq 0$ for some $w \in \mathbb{C}$, then there exists $z \in B_r(z_0)$ with $f(z) = w$.
   Hint: Recall that if we regard elements of $\pi_1(X, p)$ as pointed homotopy classes of maps $S^1 \to X$, then such a map represents the identity in $\pi_1(X, p)$ if and only if it admits a continuous extension to a map $\mathbb{D}^2 \to X$. Define X in the present case to be $\mathbb{C} \setminus \{w\}$.

   (c) Prove the Fundamental Theorem of Algebra: every nonconstant complex polynomial has a root.
   Hint: Consider loops $\gamma(t) = Re^{2\pi i t}$ with $R > 0$ large.

   (d) $(*)$ We call $z_0 \in \Omega$ an *isolated zero* of $f : \Omega \to \mathbb{C}$ if $f(z_0) = 0$ but $0 \notin f(\mathcal{U} \setminus \{z_0\})$ for some neighborhood $\mathcal{U} \subset \Omega$ of z_0. Let us say that such a zero has *order* $k \in \mathbb{Z}$ if $\operatorname{wind}(f \circ \gamma_\epsilon; 0) = k$ for $\gamma_\epsilon(t) = z_0 + \epsilon e^{2\pi i t}$ and $\epsilon > 0$ small (recall from part (a) that this does not depend on the choice of ϵ if it is small enough). Show that if $k \neq 0$, then for any neighborhood $\mathcal{U} \subset \Omega$ of z_0, there exists $\delta > 0$ such that every continuous function $g : \Omega \to \mathbb{C}$ satisfying $|f - g| < \delta$ everywhere has a zero somewhere in $\mathcal{U}$.

(e) Find an example of the situation in part (d) with $k = 0$ such that f admits arbitrarily close perturbations g that have no zeroes in some fixed neighborhood of $\mathcal{U}$.
Hint: Write f as a continuous function of x and y where $x + iy \in \Omega$. You will not be able to find an example for which f is analytic—they do not exist!

General advice: Throughout this problem, it is important to remember that $\mathbb{C} \setminus \{w\}$ is homotopy equivalent to S^1 for every $w \in \mathbb{C}$. Thus all questions about $\pi_1(\mathbb{C} \setminus \{w\})$ can be reduced to questions about $\pi_1(S^1)$.

2. For each of the following spaces X and subspaces $A \subset X$, determine whether A is a retract or a deformation retract of X, or neither. Justify your answer in each case by either describing a (deformation) retraction or saying something about fundamental groups.

   (a) $A = S^1$ in $X = \mathbb{D}^2$

   (b) $(*)$ $A = S^1 \times \{\text{pt}\}$ in $X = S^1 \times S^1$

   (c) $A = \{x_0\}$ in $X = \big([0,1] \times \{0\}\big) \cup \big(\{0\} \times [0,1]\big) \cup \big(\bigcup_{n\in\mathbb{N}} \{2^{-n}\} \times [0,1]\big)$, where $x_0 \in (0,1)$

   (d) $A = \big(S^1 \times \{y\}\big) \cup \big(\{x\} \times S^1\big)$ in $X = \big(S^1 \times S^1\big) \setminus \{(x_0, y_0)\}$ with $x_0 \neq x$ and $y_0 \neq y$

3. We can regard $\pi_1(X, p)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, \text{pt}) \to (X, p)$. Let $[S^1, X]$ denote the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. (The elements of $[S^1, X]$ are called *free homotopy classes of loops* in X). There is a natural map

$$F : \pi_1(X, p) \to [S^1, X]$$

   defined by ignoring basepoints. Prove:

   (a) F is surjective if X is path-connected.

   (b) $(*)$ $F([\alpha]) = F([\beta])$ if and only if $[\alpha]$ and $[\beta]$ are conjugate in $\pi_1(X, p)$.
   Hint: If $H : [0,1] \times [0,1] \to X$ is a homotopy with $H(0,\cdot) = \alpha$ and $H(1,\cdot) = \beta$, and $t_0 \in S^1$ is the basepoint in S^1, then $\gamma := H(\cdot, t_0)$ is also a loop based at p. Compare α and $\gamma \cdot \beta \cdot \gamma^{-1}$.

   The conclusion is that if X is path-connected, F induces a bijection between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$. In particular, $\pi_1(X) \cong [S^1, X]$ whenever $\pi_1(X)$ is abelian.

4. Here is a useful fact from linear algebra known as *polar decomposition*: every invertible real matrix $\mathbf{A} \in \text{GL}(n, \mathbb{R})$ can be written as $\mathbf{PR}$, where $\mathbf{R}$ is orthogonal and $\mathbf{P}$ is symmetric positive-definite. To see this, notice that $\mathbf{AA}^T$ is always symmetric and positive-definite, thus it can be written as $\mathbf{M\Lambda M}^T$ for some orthogonal $\mathbf{M}$ and diagonal $\mathbf{\Lambda}$ with positive entries, making it possible to define powers $(\mathbf{AA}^T)^p = \mathbf{M\Lambda}^p\mathbf{M}^T$ for every $p \in \mathbb{R}$. Then defining $\mathbf{P} := (\mathbf{AA}^T)^{1/2}$, it is not hard to verify that $\mathbf{R} := \mathbf{P}^{-1}\mathbf{A}$ is orthogonal.

   (a) Use polar decomposition to show that the group $\{\mathbf{A} \in \text{GL}(n, \mathbb{R}) \mid \det(\mathbf{A}) > 0\}$ admits a deformation retraction to the special orthogonal group $\text{SO}(n)$ for every $n \in \mathbb{N}$.[1]

   (b) Identifying S^1 with the quotient group $\mathbb{R}/\mathbb{Z}$, show that every loop $\mathbf{A} : S^1 \to \text{GL}(2, \mathbb{R})$ passing through the identity matrix is homotopic in $\text{GL}(2, \mathbb{R})$ to a loop of rotations

$$\mathbf{A}(t) = \begin{pmatrix} \cos(2\pi kt) & -\sin(2\pi kt) \\ \sin(2\pi kt) & \cos(2\pi kt) \end{pmatrix}$$

   for some $k \in \mathbb{Z}$, and k is uniquely determined by $\mathbf{A} : S^1 \to \text{GL}(2, \mathbb{R})$.
   Hint: What is $\text{SO}(2)$ homeomorphic to?

5. $(*)$ For any topological space X, the space $CX = (X \times [0,1])/(X \times \{1\})$ is called the *cone over* X. We write $[x, t]$ for the equivalence class in CX of the element (x, t) of $X \times [0,1]$. Let $s = [x, 1]$ (for all x in X) denote the "summit" of the cone. Show that CX is contractible, and X is a deformation retract of $CX \setminus \{s\}$.

[1] Here we assume $\text{GL}(n, \mathbb{R})$ carries its natural topology as an open subset of the space of all real n-by-n matrices (a vector space isomorphic to $\mathbb{R}^{n^2}$).