Course description and syllabus

General information

Instructors: Prof. Chris Wendl (lectures)
HU Institute for Mathematics (Rudower Chaussee 25), Room 1.301
`wendl@math.hu-berlin.de`
Office hour: Wednesdays 15:00–16:00

Dr. Felix Schmäschke (problem classes)
HU Institute for Mathematics (Rudower Chaussee 25), Room 1.303
`felix.schmaeschke@math.hu-berlin.de`
Office hour: Wednesdays 14:00–15:00

Course webpage: `http://www.mathematik.hu-berlin.de/~wendl/Sommer2017/Topologie1/`

Lectures: Wednesdays 9:00–11:00 in 1-0311 (Rudower Chaussee 26)
Thursdays 9:00–11:00 in 1.115 (Rudower Chaussee 25)

Problem classes: Wednesdays 11:00–13:00 in 1.013 (Rudower Chaussee 25)

Language: The course can be offered in German or English depending on the preferences of the students.

Prerequisites: Contents of the HU's courses *Analysis I* and *II*, *Lineare Algebra und Analytische Geometrie I* and *II*, and the algebraic content of *Algebra und Funktionentheorie*

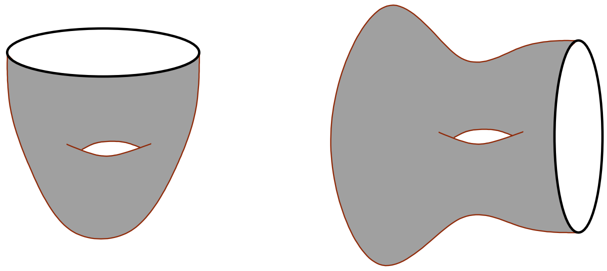
Students will be assumed to be comfortable with the theory of metric spaces from first-year analysis and with the basic notions of groups, rings and fields.

Short description

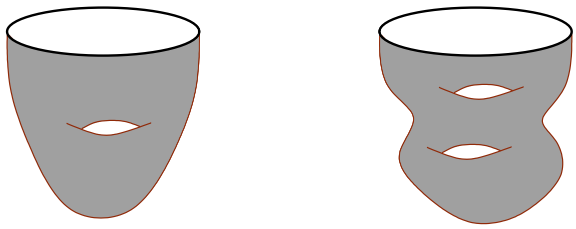
An introduction to topology with emphasis on geometric applications: metric and topological spaces, separation axioms, compactness and connectedness, the fundamental group, homotopy invariance, Seifert-van Kampen theorem, covering spaces. Basics of differential topology: topological and smooth manifolds, path-connectedness, sketch of Sard's theorem, mapping degree. Introduction to homology theory: simplicial and cell complexes, singular homology and cohomology, long exact sequences, basic computations, the Brouwer fixed point theorem, cellular homology.

Detailed description

Topological spaces are the natural setting for discussing the notion of continuity, hence topology is the study of properties that are preserved under continuous maps with continuous inverses (homeomorphisms). In other words, topology considers these two objects to be equivalent:



while developing methods to prove that these two are not:



The course will begin with the axioms of a topological space and their relation to the special case of metric spaces, generalizing familiar notions such as compactness and connectedness, and delving into less familiar territory with the standard separation axioms (e.g. the Hausdorff property). This portion of the subject is known as "general" (or "point-set") topology, and it allows for a number of pathological phenomena that are important to consider in infinite-dimensional analysis, though usually of less interest in geometry.

After this initial flirtation with frightening generality, the rest of the course will focus on spaces of geometric interest and will consider various ways to interpret and answer the question, "how many holes are there in this space?" The first of these is the *fundamental group*, an algebraic object that can be associated to every topological space and measures the failure of continuous loops in that space to be fillable by disks. Another essential notion in this context is homotopy, i.e. continuous deformations of continuous maps, and we will prove that the fundamental group is invariant not just under homeomorphism but also under the much more flexible notion of *homotopy equivalence*. We then introduce two useful tools for computing the fundamental group: the Seifert-van Kampen theorem, and the theory of covering spaces, each of which reveals intricate relationships between topology and group theory.

We will next spend a couple of weeks on the slightly more specialized subject of *differential* topology, in which general topological spaces are replaced by *smooth n-dimensional manifolds* and continuous maps are required to be infinitely differentiable. In this setting, Sard's theorem on the abundance of regular values for smooth functions leads to the definition of the *degree* $\deg(f) \in \mathbb{Z}$ for any continuous map $f : M \to N$ between closed and connected smooth manifolds of the same dimension; intuitively, $\deg(f)$ is a homotopy-invariant answer to the question, "for a generic point $p \in N$, how many points are there in $f^{-1}(p)$?".

The final portion of the course will introduce the much larger subject of homology and cohomology, yet another way of counting holes (of various dimensions) in topological spaces. After some preliminary motivation via simplicial complexes, we will define the singular homology and cohomology groups and the homomorphisms induced on them by continuous maps, together with requisite notions from homological algebra such as chain complexes, exact sequences and "diagram chasing". This enables us to prove the homotopy invariance of singular homology and to compute it for a few simple examples, e.g. for spheres, leading to a proof of the Brouwer fixed point theorem. If time permits, we will also discuss CW-complexes and give a sketch of cellular homology, a much more powerful computational tool.

Literature

The course will not follow any particular book, but there are many good books on the subject, in particular:

- Allen Hatcher, *Algebraic Topology*, Cambridge University Press 2002
  (also freely downloadable from the author's homepage:
  `https://www.math.cornell.edu/~hatcher/AT/ATpage.html`)

Hatcher's book covers most of what we will talk about in both this course and in Topology II the following semester, though sometimes from a slightly different perspective. It does not say much about differential topology, but for the portions of it that we will cover, there is another excellent resource:

- John Milnor, *Topology from the Differentiable Viewpoint*, Princeton University Press 1997

I can also highly recommend the following algebraic topology books, especially for material that we will cover next semester in Topology II:

- Glen Bredon, *Topology and Geometry*, Springer GTM 1993
  (online access available via the HU library)
- James W. Vick, *Homology Theory*, Springer GTM 1994
  (online access available via the HU library)
- R. Stöcker und H. Zieschang, *Algebraische Topologie - Eine Einführung*, Teubner 1994
  (available in the HU library, Freihandbestand)
- Jänich, *Topologie*, Springer 2008
  (less relevant to this course, but amusingly written — online access available via the HU library)

For a more detailed treatment of the important technical results in differential topology (some of which we will mention but not prove in this course), one of the classic references is:

- Morris W. Hirsch, *Differential Topology*, Springer GTM 1976
  (online access available via the HU library)

The following book is a standard reference for an in-depth treatment of *general topology*, i.e. the portion of the subject that deals with unrestricted classes of topological spaces, rather than algebraic invariants or applications to geometry. (Here I'm using the word "reference" to mean "I would never recommend this as a textbook for a course, but I feel reassured to know that it is on my bookshelf".)

- John L. Kelley, *General Topology*, Springer GTM 1975

Exam and problem sets

Grades in the course will be determined by a three-hour **written exam** shortly after the end of the semester (with a resit option shortly before the beginning of the following semester). Books and notes may be consulted during the exam.

There will be graded **problem sets** handed out every Wednesday and due before the problem class on the following Wednesday (the solutions will be discussed in the problem class).

Midway through the semester there will also be a **take-home midterm**. This is like a problem set, but more serious: you will have two weeks to work on it and will be required to write up your solutions alone, not collaboratively.

Your cumulative grades on the homework and the take-home midterm can be used to boost your final exam grade as follows:

- $\geq 50\%$ on homework **or** $\geq 75\%$ on midterm = (2,0 $\rightarrow$ 1,7 or 1,7 $\rightarrow$ 1,3 etc.)
- $\geq 50\%$ on homework **and** $\geq 75\%$ on midterm = (2,3 $\rightarrow$ 1,7 or 2,0 $\rightarrow$ 1,3 etc.)

Syllabus

The course is divided roughly into four units:

I. **General topology** (weeks 1–2)

II. **The fundamental group** (weeks 3–8)

III. **Differential topology** (weeks 8–10)

IV. **Homology** (weeks 10–14)

The following week-by-week plan for the lectures is tentative and subject to change.

1. General introduction and motivation, metric spaces, continuity, homeomorphisms.
2. Topological spaces, standard constructions (products, disjoint union, quotients, gluing), compactness and connectedness, separation axioms.
3. Paths, path-connectedness, the fundamental group $\pi_1(X)$.
4. Homotopy, retraction, further properties of $\pi_1(X)$.
5. Finitely-presented groups and relations, the Seifert-van Kampen theorem.
6. Introduction to covering spaces, the lifting theorem.
7. The universal cover, deck transformations, regular/normal covers and relation to $\pi_1(X)$.
8. Introduction to the higher homotopy groups $\pi_k(X)$, topological groups, topological and smooth manifolds.
9. Critical points and Sard's theorem, implicit function theorem, mapping degree modulo 2.
10. Orientations and the integer-valued mapping degree, simplicial complexes and sketch of simplicial homology.
11. Singular homology $H_*(X)$ and cohomology $H^*(X)$, chain maps, exact sequences, the long exact sequence of a pair.
12. Homotopy invariance of homology, the Mayer-Vietoris sequence, computation of $H_0(X)$ and $H_1(X)$.
13. Computation of $H_*(S^n)$, the Brouwer fixed point theorem, introduction to CW-complexes and cellular homology.
14. (further topics to be decided if time permits)

Preview: Topics to be covered in Topology II (Winter 2017–2018)

Definitely:
Axiomatic homology and cohomology (the Eilenberg-Steenrod axioms), equivalence of cellular and singular homology, cup product on cohomology, Künneth formula, universal coefficient theorem, fundamental classes and Poincaré duality, intersection theory, higher homotopy groups, Serre fibrations and the homotopy exact sequence, Hurewicz homomorphism $\pi_k(X) \to H_k(X)$, Whitehead's theorem on weak homotopy equivalences.

Optionally:
Lefschetz fixed point theorem, differential forms and de Rham's theorem, bundles and classifying spaces, obstruction theory, bordism groups, characteristic classes, exotic spheres.