Topology I
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PROBLEM SET 8
Due: 3.07.2018

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

1. Prove each of the following, assuming $p : Y \to X$ is a covering map with X and Y both path-connected.

   (a) If $\mathcal{U} \subset X$ is evenly covered, then so is every subset of $\mathcal{U}$.

   (b) The map $p : Y \to X$ is open, i.e. it sends open subsets of Y to open subsets of X.

   (c) For every $x \in X$, $f^{-1}(x)$ is a discrete subset of Y.[1]

   (d) If Y is compact, then X is also compact and $\deg(p) < \infty$.

   (e) $(*)$ The map $p : Y \to X$ is proper[2] if and only if $\deg(p) < \infty$.
   Hint: Showing that properness implies finite degree is easy. For the converse, given a compact set $K \subset X$ and an open cover $f^{-1}(K) \subset \bigcup_\alpha \mathcal{U}_\alpha$, it suffices to find a finite cover of $f^{-1}(K)$ by open sets such that each is contained in some $\mathcal{U}_\alpha$. (Why?) Start by showing that K can be covered by a finite collection of open neighborhoods which are evenly covered and small enough so that their (finitely many!) lifts to Y are each contained in some $\mathcal{U}_\alpha$.

   (f) Deduce from the above that the converse of part (d) also holds: if $\deg(p) < \infty$ and X is compact, then Y is also compact.

2. Assume $p : Y \to X$ is a covering map and X is path-connected.

   (a) $(*)$ Use the lifting theorem to show that for any two points $x, y \in X$, lifting paths from x to y associates to each such path γ a bijection $\rho_\gamma : p^{-1}(x) \to p^{-1}(y)$, which depends only on the homotopy class of γ (with fixed end points).

   (b) Writing $J := p^{-1}(x)$ and applying part (a) in the case $x = y$ gives a map
   $$\rho : \pi_1(X, x) \to S(J) : [\gamma] \mapsto \rho_\gamma$$
   where $S(J)$ is the group of all bijections $J \to J$.[3] Show that this map is a group homomorphism.

   (c) $(*)$ Write down the homomorphism $\rho : \pi_1(X, x) \to S(J)$ explicitly for the space $(X, x) = (\mathbb{C}^* := \mathbb{C}\backslash\{0\}, 1)$ with covering map $p : \mathbb{C} \to \mathbb{C}^* : z \mapsto e^z$.

3. (a) Show that every covering map of degree 2 is regular.
   Hint: There is an algebraic way to solve this problem, but a more direct approach is also possible.

   (b) Prove that every covering map of the torus $\mathbb{T}^2 = S^1 \times S^1$ is regular.

   (c) Find all subgroups of $\mathbb{Z}^2$ with index 2.
   Hint: Given such a subgroup $H \subset \mathbb{Z}^2$, consider the images of the two generators $e_1 := (1, 0)$ and $e_2 := (0, 1)$ of $\mathbb{Z}^2$ under the quotient homomorphism $\mathbb{Z}^2 \to \mathbb{Z}^2/H \cong \mathbb{Z}_2$. Show that there are exactly three possibilities, depending on whether each of e_1 or e_2 represents the trivial or nontrivial element in the quotient.

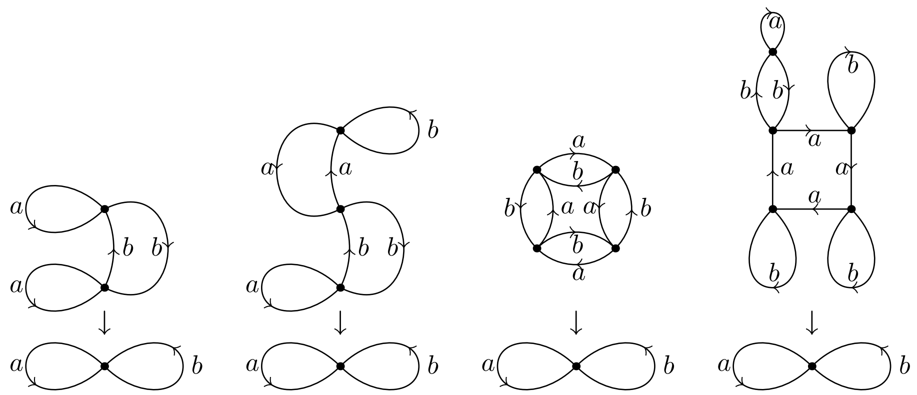
[1] We say that a subset A in a space X is *discrete* if the subspace topology induced by X on A is the same as the discrete topology.

[2] A map $f : X \to Y$ is said to be *proper* (*eigentlich*) if for every compact subset $K \subset Y$, $f^{-1}(K) \subset X$ is also compact.

[3] Notice that if J is a set of n elements, $S(J)$ is isomorphic to the *symmetric group* S_n.

(d) ($*$) Deduce from part (c) that up to isomorphism of covers, $\mathbb{T}^2$ admits exactly three distinct covering maps with degree 2, and write them down explicitly.
Hint: You may have to take an educated guess as to what the covering spaces should be, but notice that part (c) tells you what their fundamental groups are.

4. Convince yourself that the maps depicted in the figure below are covers, and determine their deck transformation groups. Which ones are regular?



5. ($*$) Prove that for every path-connected topological group G, $\pi_1(G)$ is abelian.
Hint: Fix the identity element $e \in G$ as a base point, and let e also denote the constant path at e. Given loops α, β in G based at e, it is easy to construct a homotopy from $\alpha \cdot e$ to $e \cdot \alpha$ and also a homotopy from $e \cdot \beta$ to $\beta \cdot e$. Can you use the group structure to turn these into a homotopy between $\alpha \cdot \beta$ and $\beta \cdot \alpha$?

6. The *Prüfer surface* is an example of a space that would be a connected 2-dimensional manifold if we did not require manifolds to be second countable. It is defined as follows: let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, and associate to each $a \in \mathbb{R}$ a copy of the plane $X_a := \mathbb{R}^2$. The Prüfer surface is then

$$\Sigma := \mathbb{H} \amalg \left(\coprod_{a \in \mathbb{R}} X_a \right) \Big/ \sim$$

where the equivalence relation identifies each point $(x, y) \in X_a$ for $y > 0$ with the point $(a + yx, y) \in \mathbb{H}$. Notice that $\mathbb{H}$ and X_a for each $a \in \mathbb{R}$ can be regarded naturally as subspaces of Σ.

(a) Prove that Σ is Hausdorff.

(b) Prove that Σ is path-connected.

(c) Prove that every point in Σ has a neighborhood homeomorphic to $\mathbb{R}^2$.

(d) Prove that a second countable space can never contain an uncountable discrete subset. Then find an uncountable discrete subset of Σ.