Topology I
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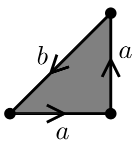
PROBLEM SET 9
Due: 10.07.2018

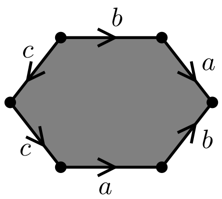
Instructions

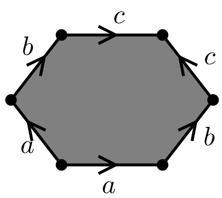
Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

1. Each of the following pictures defines a topological space by identifying all vertices of the polygon to a single point and identifying any pairs of edges with matching letters via a homeomorphism that matches the arrows. Determine whether each space is (i) a 2-manifold (without boundary), (ii) a 2-manifold with boundary, or (iii) neither.
Hint: There are three kinds of points in each space, namely those that appear in the interior of the polygon, those that appear on an edge but not on a vertex, and the one point represented by all vertices. You need to understand what small neighborhoods of each of these points look like.

(a)


(b)


(c)


(d) For any cases in parts (a)–(c) where the space is a manifold or a manifold with boundary, describe it in terms of familiar surfaces such as the disk, the Möbius band, the sphere, the torus, the projective plane, and connected sums of these. Is it orientable?

Remark: You can answer parts (a)–(c) without doing part (d), but if you can see how to do part (d), then it's a good way to verify whether your answers to parts (a)–(c) were correct. In some cases, you might also be able to compute the fundamental groups of the spaces in part (d) and compare the result with whatever the polygon pictures tell you.

2. On Problem Set 7 we considered the space $\Sigma_{g,m}$, defined by cutting the interiors of $m \geqslant 0$ disjoint disks out of the oriented surface Σ_g of genus $g \geqslant 0$.

(a) (∗) Prove that every compact, orientable, connected surface with boundary is homeomorphic to $\Sigma_{g,m}$ for some values of $g, m \geqslant 0$.
Hint: If Σ is a compact 2-manifold, then $\partial\Sigma$ is a closed 1-manifold, and we classified all of the latter. With this knowledge, there is a cheap trick by which you can turn any compact surface with boundary into a closed surface, and then apply what you have learned about the classification of closed surfaces. Don't forget to keep track of orientations.

(b) (∗) Prove that $\Sigma_{g,m}$ is homeomorphic to $\Sigma_{h,n}$ if and only if $g = h$ and $m = n$.

3. (∗) Recall that if Σ is a surface with boundary, the boundary $\partial\Sigma$ is defined as the set of all points $p \in \Sigma$ such that some chart $\varphi : \mathcal{U} \xrightarrow{\cong} \Omega \subset \mathbb{H}^2$ defined on a neighborhood $\mathcal{U} \subset \Sigma$ of p satisfies $\varphi(p) \in \partial\mathbb{H}^2$. Here $\mathbb{H}^2 := [0, \infty) \times \mathbb{R} \subset \mathbb{R}^2$, $\partial\mathbb{H}^2 := \{0\} \times \mathbb{R} \subset \mathbb{H}^2$, and Ω is an open subset of $\mathbb{H}^2$. One can analogously define $p \in \Sigma$ to be an *interior point* of Σ of some chart maps it to $\mathbb{H}^2 \backslash \partial\mathbb{H}^2$. Prove that no point on $\partial\Sigma$ is also an interior point of Σ.
Hint: If you have two charts defined near p such that one sends p to $\partial\mathbb{H}^2$ while the other sends it to $\mathbb{H}^2 \backslash \partial\mathbb{H}^2$, then a transition map relating these two charts maps some neighborhood in $\mathbb{H}^2$ of a point

$x \in \mathbb{H}^2 \backslash \partial\mathbb{H}^2$ to a neighborhood in $\mathbb{H}^2$ of a point $y \in \partial\mathbb{H}^2$. What happens to this homeomorphism if you remove the points x and y? Think about the fundamental group.
Remark: A similar result is true for topological manifolds of arbitrary dimension, but you do not yet have enough tools at your disposal to prove this. A proof using singular homology will be possible before the end of the semester.

4. This problem requires the following notions from differential topology, some of which have been mentioned in lecture:

   - A topological manifold M is called a *smooth manifold* if it is endowed with a maximal collection of charts $\{\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\cong} \Omega_\alpha\}_{\alpha \in J}$ such that $M = \bigcup_{\alpha \in J} \mathcal{U}_\alpha$ and the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are C^∞ wherever they are defined. We call the charts φ_α in this collection *smooth charts*.
   - For two smooth manifolds M and N, a map $f : M \to N$ is called *smooth* if for every pair of smooth charts ψ_β on N and φ_α on M, the map $f_{\beta\alpha} := \psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is C^∞ wherever it is defined. (In other words, f is "C^∞ in local coordinates".)
   - For an open subset $\Omega \subset \mathbb{H}^n$ and a C^∞ map $f : \Omega \to \mathbb{R}^m$, a point $y \in \mathbb{R}^m$ is called a *regular value* of f if the derivative $df(x) : \mathbb{R}^n \to \mathbb{R}^m$ is surjective for every $x \in f^{-1}(y)$.
   - For $f : M \to N$ a smooth map between smooth manifolds, a point $q \in N$ is a *regular value* of f if for all charts φ_α on M and ψ_β on N such that q is in the domain of ψ_β, $\psi_\beta(q)$ is a regular value of $f_{\beta\alpha}$. (In other words, q is a "regular value of f in local coordinates".)

   An easy corollary of the usual implicit function theorem then states that if M is a smooth m-manifold without boundary, N is a smooth n-manifold and $f : M \to N$ is a smooth map that has $q \in N$ as a regular value, the preimage $f^{-1}(q) \subset M$ is a smooth submanifold[1] of dimension $m - n$. If M has boundary, then one should assume additionally that q is a regular value of the restricted map $f|_{\partial M} : \partial M \to N$, and the conclusion is then that $Q := f^{-1}(q)$ is a smooth manifold of dimension $m - n$ with boundary $\partial Q = Q \cap \partial M$.

   Hopefully you don't find any of the above statements implausible, but I shall ask you in any case to take them on faith for now, in addition to the following perturbation result: if M and N are compact smooth manifolds, $q \in N$ and $f : M \to N$ is continuous, then every neighborhood of f in $C(M, N)$ with the compact-open topology contains a smooth map $f_\epsilon : M \to N$ for which q is a regular value of both f_ϵ and $f_\epsilon|_{\partial M}$. Moreover, if $f|_{\partial M}$ is already smooth and has q as a regular value, then the perturbation can be chosen such that $f_\epsilon|_{\partial M} = f|_{\partial M}$.

   If you take all of this as given, then you can use it to define something quite beautiful. Assume M and N are closed connected smooth manifolds of the same dimension n. Then for any smooth map $f : M \to N$ with regular value $q \in N$, the implicit function theorem implies that $f^{-1}(q)$ is a compact 0-manifold, i.e. a finite set of points. Define the *mod 2 mapping degree* $\deg_2(f) \in \mathbb{Z}_2$ of f by

   $$\deg_2(f) := |f^{-1}(q)| \pmod 2.$$

   (a) (∗) Prove that $\deg_2(f)$ depends only on the homotopy class of $f : M \to N$.
   Hint: If you have a homotopy $H : I \times M \to N$ between two maps, perturb it as necessary and look at $H^{-1}(q)$. Use the classification of compact 1-manifolds.
   Remark: One can show with a little more effort that $\deg_2(f)$ also does not depend on the choice of the point q, and moreover, it has a well-defined extension to continuous (but not necessarily smooth) maps $f : M \to N$, defined by setting $\deg_2(f) := \deg_2(f_\epsilon)$ for any sufficiently close smooth perturbation f_ϵ.

   (b) (∗) Prove that every continuous map $f : S^2 \to S^2$ homotopic to the identity is surjective.

   (c) What goes wrong with this discussion of we allow M to be a noncompact manifold? Describe two homotopic maps $f, g : \mathbb{R} \to S^1$ for which $\deg_2(f) \neq \deg_2(g)$.

   (d) Prove that if $n > m$, every continuous map $S^m \to S^n$ is homotopic to a constant map.
   Hint: What does it mean for a point $q \in S^n$ to be a regular value of $f : S^m \to S^n$ if $n > m$?

[1] A subset $Y \subset M$ of a smooth manifold M is called a *smooth submanifold* if it is admits the structure of a smooth manifold such that the inclusion $Y \hookrightarrow M$ is smooth.