Topology I
C. Wendl

Humboldt-Universität zu Berlin
Summer Semester 2018

Course description and syllabus

General information

Instructor: Prof. Chris Wendl
HU Institute for Mathematics (Rudower Chaussee 25), Room 1.301
`wendl@math.hu-berlin.de`
Office hour: Thursdays 14:00–15:00

Course webpage: `http://www.mathematik.hu-berlin.de/~wendl/Sommer2018/Topologie1/`

Lectures: Tuesdays 13:00–15:00 in 1-0311 (Rudower Chaussee 26)
Fridays 11:00–13:00 in 1.013 (Rudower Chaussee 25)

Problem classes: Tuesdays 15:00–17:00 in 1.013 (Rudower Chaussee 25)

Language: The course can be offered in German or English depending on the preferences of the students.

Prerequisites: Contents of the HU's courses *Analysis I* and *II*, *Lineare Algebra und Analytische Geometrie I* and *II*, and the algebraic content of *Algebra und Funktionentheorie*

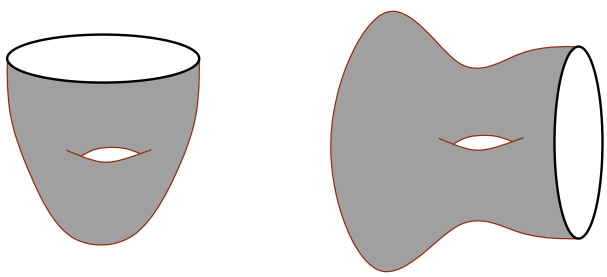
Students will be assumed to be comfortable with the theory of metric spaces from first-year analysis and with the basic notions of groups, rings and fields.

Short description

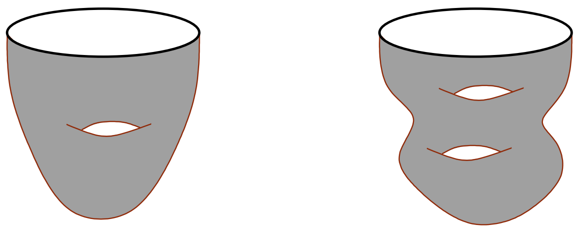
An introduction to topology with emphasis on geometric applications: metric and topological spaces, separation axioms, compactness and connectedness, the fundamental group, homotopy invariance, Seifert-van Kampen theorem, covering spaces. Introduction to homology theory: topological manifolds, simplicial complexes and triangulations, singular homology, long exact sequences, the Brouwer fixed point theorem, sketch of cellular homology.

Detailed description

Topological spaces are the natural setting for discussing the notion of continuity, hence topology is the study of properties that are preserved under continuous maps with continuous inverses (homeomorphisms). In other words, topology considers these two objects to be equivalent:



while developing methods to prove that these two are not:



The course will begin with the axioms of a topological space and their relation to the special case of metric spaces, generalizing familiar notions such as compactness and connectedness, and delving into less familiar territory with the standard countability and separation axioms (e.g. the Hausdorff property). This portion of the subject is known as "general" (or "point-set") topology, and it allows for a number of pathological phenomena that are important to consider in infinite-dimensional analysis, though usually of less interest in geometry.

After this initial flirtation with frightening generality, the rest of the course will focus on spaces of geometric interest and will consider various ways to interpret and answer the question, "how many holes are there in this space?" The first of these is the *fundamental group*, an algebraic object that can be associated to every topological space and measures the failure of continuous loops in that space to be fillable by disks. Another essential notion in this context is homotopy, i.e. continuous deformations of continuous maps, and we will prove that the fundamental group is invariant not just under homeomorphism but also under the much more flexible notion of *homotopy equivalence*. We then introduce two useful tools for computing the fundamental group: the Seifert-van Kampen theorem, and the theory of covering spaces, each of which reveals intricate relationships between topology and group theory.

The final portion of the course will introduce the much larger subject of homology, yet another way of counting holes (of various dimensions) in topological spaces. After some preliminary motivation via bordism theory and simplicial complexes, we will define the singular homology groups and the homomorphisms induced on them by continuous maps, together with requisite notions from homological algebra such as chain complexes, exact sequences and "diagram chasing". This enables us to compute singular homology for a few simple examples, e.g. for spheres, leading to a proof of the Brouwer fixed point theorem. The focus throughout will be on understanding the geometric meaning of the homology groups. If time permits, we will also discuss CW-complexes and give a sketch of cellular homology, a much more powerful computational tool.

Syllabus

The course is divided roughly into three units:

I. **General topology** (weeks 1–3)

II. **The fundamental group** (weeks 4–9)

III. **Homology** (weeks 10–14)

The following week-by-week plan for the lectures is tentative and subject to change.

1. General introduction and motivation, metric spaces, continuity and sequential continuity.
2. Axioms of a topological space, bases and subbases, standard examples and constructions (subspaces, products, disjoint unions), countability axioms, convergence of nets.
3. Compactness and sequential compactness, separation axioms, Tychonoff's theorem.

4. Connected and path-connected spaces, quotient topologies, paths and homotopy, fundamental group.
5. Simply connected spaces, retractions and deformation retractions, homotopy equivalence.
6. Cone and suspension, group presentations, statement of the Seifert-van Kampen theorem.
7. Applications and proof of the Seifert-van Kampen theorem.
8. Covering spaces, lifting theorem, deck transformations.
9. The Galois correspondence, the universal cover, topological groups and group actions
10. Topological manifolds, connected sum, triangulations, sketch of the classification of closed surfaces.
11. Bordism groups, simplicial complexes and simplicial homology, basic homological algebra.
12. Singular homology, subdivision and homotopy invariance, short and long exact sequences
13. Excision, the homology of the sphere, the Brouwer fixed point theorem, mapping degree, introduction to cellular homology
14. (further topics to be decided if time permits)

Literature

The course will not follow any one particular book, but there are several good books that I can recommend. Almost all topics in the initial point-set topology unit of this course (with the notable exception of net convervence, for which I will provide lecture notes) are well covered in the book by Jänich, which also has the advantage of being amusingly written. The original German version has gone through several new editions since the English translation was published in 1984, but the revisions have been minor, so the two are still essentially the same book:

- Jänich, *Topologie*, 8. Auflage, Springer 2005
  (online access available via the HU library)
- Jänich, *Topology*, translated from the German by Silvio Levy, Springer 1984
  (available in the HU library, Freihandbestand)

Starting from week four and continuing into the followup course next semester, we will focus on *algebraic* topology, for which the most popular book is:

- Allen Hatcher, *Algebraic Topology*, Cambridge University Press 2002
  (also freely downloadable from the author's homepage:
  `https://www.math.cornell.edu/~hatcher/AT/ATpage.html`)

I can also highly recommend the following algebraic topology books, especially for material that we will cover next semester in Topology II:

- Glen Bredon, *Topology and Geometry*, Springer GTM 1993
  (online access available via the HU library)
- James W. Vick, *Homology Theory*, Springer GTM 1994
  (online access available via the HU library)
- R. Stöcker und H. Zieschang, *Algebraische Topologie - Eine Einführung*, Teubner 1994
  (available in the HU library, Freihandbestand)

Unfortunately there is not enough time in this course to discuss the closely related subject of *differential* topology, but I would nonetheless like to mention two books on this topic that might serve as interesting reading in your spare time. The first, in particular, is a classic that everyone should read at some point (and it is short!):

- John Milnor, *Topology from the Differentiable Viewpoint*, Princeton University Press 1997
- Morris W. Hirsch, *Differential Topology*, Springer GTM 1976
  (online access available via the HU library)

Finally, here is a standard reference for an in-depth treatment of *general* topology, i.e. the portion of the subject that deals with unrestricted classes of topological spaces, rather than algebraic invariants or applications to geometry. (Here I'm using the word "reference" to mean "I would never recommend this as a textbook for a course, but I feel reassured to know that it is on my bookshelf".)

- John L. Kelley, *General Topology*, Springer GTM 1975

Exam and problem sets

Grades in the course will be determined by a three-hour **written exam** shortly after the end of the semester (with a resit option shortly before the beginning of the following semester). Books and notes may be consulted during the exam.

There will be graded **problem sets** handed out every Tuesday and due before the problem class on the following Tuesday (the solutions will be discussed in the problem class).

Midway through the semester there will also be a **take-home midterm**. This is like a problem set, but you will have two weeks to work on it.

Please note: for all assignments (including the take-home midterm) in this class, you may discuss the problems with your fellow students, but you must **write up your solutions alone** (without help).

Your cumulative grades on the homework and the take-home midterm can be used to boost your final exam grade as follows:

- $\geq 50\%$ on homework **or** $\geq 75\%$ on midterm = (2,0 $\rightarrow$ 1,7 or 1,7 $\rightarrow$ 1,3 etc.)
- $\geq 50\%$ on homework **and** $\geq 75\%$ on midterm = (2,3 $\rightarrow$ 1,7 or 2,0 $\rightarrow$ 1,3 etc.)

Preview: Topics to be covered in Topology II (Winter 2018–2019)

Categories and functors, Mayer-Vietoris and other exact sequences, homology with coefficients, cohomology, the Eilenberg-Steenrod axioms, alternatives to singular (co-)homology (the Čech and Alexander-Spanier theories), direct and inverse limits, computation of cellular (co-)homology via the axioms, universal coefficient theorems, Lefschetz fixed point theorem, Künneth formula, cross and cup products, fundamental classes and Poincaré duality, intersection theory, higher homotopy groups