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Differentialgeometrie II
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Problem Set 5

To be discussed: 1.06.2022

Problem 1
For the smooth $\mathbb{Z}_2$-action on $\mathbb{R}^n$ defined via the antipodal map, prove:

(a) $\mathbb{R}^2/\mathbb{Z}_2$ is homeomorphic to $\mathbb{R}^2$.

(b) The quotient of $\mathbb{R}^3\backslash\{0\}$ by $\mathbb{Z}_2$ can be endowed with a smooth manifold structure for which the quotient projection is smooth, but $\mathbb{R}^3/\mathbb{Z}_2$ cannot.
Hint: $[0] \in \mathbb{R}^3/\mathbb{Z}_2$ *has a neighborhood bounded by a surface diffeomorphic to* $\mathbb{RP}^2$. *Deduce that if* $\mathbb{R}^3/\mathbb{Z}_2$ *is a manifold, then* $\mathbb{RP}^2$ *must be orientable.*

Problem 2
Show that for any smooth and proper action of a Lie group G on a manifold M:

(a) The quotient M/G is Hausdorff.

(b) The stabilizer $G_p \subset G$ is compact for every $p \in M$.

Problem 3
A smooth free group action $G \times M \to M$ is called *properly discontinuous* if every $p \in M$ has a neighborhood $\mathcal{U} \subset M$ such that $g\mathcal{U} \cap \mathcal{U} = \varnothing$ for all $g \in G\backslash\{e\}$. Show that this condition holds if and only the action is proper and the group G is discrete.
Note: We are also assuming the action is free.[1]

Problem 4
Recall that for any smooth, free and proper group action $G \times M \to M$, the slice theorem endows M/G with a smooth manifold structure for which the quotient projection $\pi : M \to M/G$ is a smooth submersion. One of the important consequences of this problem will be that you rarely need to know precisely how this smooth structure is constructed, because the condition of π being a smooth submersion determines it uniquely. Prove:

(a) A map $f : M/G \to N$ to another smooth manifold N is smooth if and only if the composition $f \circ \pi : M \to N$ is smooth.
Hint: Do you remember what submersions look like in cleverly chosen coordinates?

(b) The derivative at $p \in M$ of $\pi : M \to M/G$ descends to a vector space isomorphism

$$T_pM/T_p(Gp) \xrightarrow{\cong} T_{[p]}(M/G).$$

Assume next that $H \times N \to N$ is another smooth, free and proper group action. A map $F : M \to N$ is said to *descend* to the quotient if there exists a map $f : M/G \to N/H$ such that $f([p]) = [F(p)]$ for every $p \in M$. Prove:

(c) If $G = H$ and the map $F : M \to N$ is G-equivariant, then it descends to the quotient.

(d) If $F : M \to N$ is smooth and descends to the quotient, then the induced map $f : M/G \to N/H$ is also smooth.

[1] There does not seem to be a unanimous consensus on what the term "properly discontinuous" should mean for an action that is not free.

Problem 5
The multiplicative group $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$ acts smoothly on $\mathbb{C}^{n+1}\backslash\{0\}$ via scalar multiplication, and the quotient $(\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^*$ is by definition $\mathbb{CP}^n$. Prove that this action is free and proper, and that the resulting smooth structure on $\mathbb{CP}^n$ is the same as the one you constructed via an explicit atlas in Problem Set 1 #3.

Problem 6
Given smooth, free and proper group actions $G \times M \to M$ and $H \times N \to N$, prove:

(a) The Lie group $G \times H$ acts smoothly, freely and properly on $M \times N$ by $(g, h) \cdot (p, q) := (gp, hq)$.

(b) The map $(M \times N)/(G \times H) \to (M/G) \times (N/H) : [(p, q)] \mapsto ([p], [q])$ is well defined and gives a diffeomorphism.

Problem 7
Assume that G is a Lie group with a Lie subgroup $H \subset G$, and let G/H denote the set of left cosets $\{gH \subset G \mid g \in G\}$. Prove:

(a) The right action of H on G defined by $G \times H \to G : (g, h) \mapsto gh$ is smooth, free and proper.

(b) For the smooth structure defined on G/H via the slice theorem, the map $G \times (G/H) \to G/H : (g, aH) \mapsto gaH$ defines a smooth left action of G on G/H.
Hint: Problem 6 makes $G \times (G/H)$ diffeomorphic to the quotient of $G \times G$ by a free and proper action of some product subgroup. You can therefore use Problem 4 to check the smoothness of a map defined on $G \times (G/H)$.

(c) If the subgroup $H \subset G$ is normal, then G/H has a natural Lie group structure for which the quotient projection $G \to G/H$ is a Lie group homomorphism.

Problem 8
Assume $G \times M \to M$ is a smooth proper group action such that the isotropy subgroup $G_p \subset G$ is finite for every $p \in M$. The goal of this problem is to show, via a mild generalization of the slice theorem, that G/H is then an *orbifold* of dimension $\dim M - \dim G$, a notion generalizing the concept of a manifold. In an n-dimensional orbifold, every point has a neighborhood homeomorphic to the quotient of an open subset of $\mathbb{R}^n$ by a finite group action.

(a) Show that for each $p \in M$, the orbit $Gp \subset M$ is a smooth submanifold. Unlike the case of a free action, Gp will not generally be diffeomorphic to G. What instead?

(b) Show that p admits a G_p-invariant neighborhood $\mathcal{U} \subset M$ with a G_p-invariant Riemannian metric, i.e. each $g \in G_p$ acts on $\mathcal{U}$ via an isometry.
Hint: Start with any metric, then act on it with G_p and take an average.

(c) Construct a submanifold $\Sigma \subset M$ that satisfies $\Sigma \cap Gp = \{p\}$ and $T_p\Sigma \oplus T_p(Gp) = T_pM$ and, additionally, is invariant under the action of G_p.
Hint: Use geodesics.

(d) Show that after possibly shrinking Σ to a smaller neighborhood of p, the map $\Phi : G \times \Sigma \to M : (g, q) \mapsto gq$ can be assumed to be a local diffeomorphism satisfying $g(\Sigma) \cap \Sigma = \varnothing$ for all $g \in G\backslash G_p$.

(e) Conclude that the map $\Sigma \to M/G : p \mapsto [p]$ descends to the quotient Σ/G_p as a homeomorphism onto a neighborhood of $[p]$ in M/G.

(f) Deduce that for every $p \in M$, suffiently nearby points $q \in M$ satisfy $|G_q| \leqslant |G_p|$.