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Differentialgeometrie II

SoSe 2022

Problem Set 7

To be discussed: 15.06.2022

Problem 1

In lecture we proved that if $\pi : E \to M$ has a G-bundle atlas with standard fiber G acted upon by the structure group G via left translation, then this bundle atlas determines a natural fiber-preserving right G-action on E that is free and transitive on every fiber. Extend this result as follows: if $\pi^j : E^j \to M$ for $j = 1, 2$ are two bundles with G-bundle atlases as described above, then a smooth fiber-preserving map $\Psi : E^1 \to E^2$ is a G-bundle isomorphism if and only if it is equivariant with respect to the two right G-actions.
Comment: This completes the proof that the two definitions of the term "principal G-bundle" given in lecture coincide.

Problem 2

Show that if $G \times M \to M$ is a smooth and transitive left group action, then for any $p \in M$, the map $G \to M : p \mapsto gp$ defines a principal G_p-bundle, where the stabilizer G_p acts on G by right translation.

Problem 3

Let $E \to \mathbb{CP}^n = \mathrm{Gr}_1(\mathbb{C}^{n+1})$ denote the tautological vector bundle defined in Problem Set 6 #3, which is in this case a complex line bundle. Each fiber of E is naturally a subspace of $\mathbb{C}^{n+1}$, so by restriction, the standard Hermitian inner product on $\mathbb{C}^{n+1}$ defines a bundle metric on E, i.e. a $\mathrm{U}(1)$-structure.

(a) Give an explicit description of the orthonormal frame bundle $F^{\mathrm{O}}(E) := F^{\mathrm{U}(1)}E \to \mathbb{CP}^n$, including its right $\mathrm{U}(1)$-action. To what more familiar manifold is the total space $F^{\mathrm{O}}(E)$ diffeomorphic?

(b) Prove that the bundle $E \to \mathbb{CP}^n$ is not trivial.
Note: This probably requires a bit of algebraic topology, e.g. some basic knowledge of the fundamental group. Don't use more than you have to.

(c) After you've thought through parts (a) and (b), if you have some spare time, watch the following beautiful video about the Hopf fibration:
`https://www.youtube.com/watch?v=yNpqLMpfxA8&list=PL3C690048E1531DC7&t=3s`
(This is Chapter 7 of "*Dimensions*": `http://www.dimensions-math.org/Dim_E.htm`)

Problem 4

Prove via partitions of unity that every smooth fiber bundle $\pi : E \to M$ admits a connection, and that the set of all connections on $\pi : E \to M$ naturally has the structure of an affine space. (Over what vector space?)
Hint: Think in terms of connection 1-forms $K \in \Omega^1(E, VE)$.

Problem 5

Assume $\pi : E \to M$ is a smooth fiber bundle whose fibers are compact. Prove that $\pi : E \to M$ admits a flat connection if and only if it admits a G-structure where G is a 0-dimensional Lie group.