<u>Notation</u> $f: \mathbb{R}^n \to \mathbb{R}^p$ smooth map. (without saying, it's auto smooth

i.e. $\exists\, U \subset \mathbb{R}^n : f: U \to \mathbb{R}^q$, & ~~call~~ denote $\mathrm{dom}(f)$ the domain of f,

• $df_{x_0}(u) = \boxed{\lim_{t\to 0}} \frac{f(x_0+tu) - f(x_0)}{t}$, $\forall\, u \in \mathbb{R}^n$.
at x_0

only defined in a nbh of x_0.
the size of the nbh doesn't matter.

<u>Observe</u>

(want to define some algebra doesn't care about the nbh)

$\Longrightarrow$

<u>Def</u> (Two maps) $f, g: \mathbb{R}^n \to \mathbb{R}^p$ are ~~called~~ (said to be)

<u>germ equivalent</u> at (a pt) $q \in \mathbb{R}^n$ if.

$q \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$

& $\exists$ (nbh) $U \ni q$: $f|_U = g|_U$

<u>Rmk</u> • Indeed equivalence relation: reflexive, symmetry, transitivity..

<u>~~Notation~~ Def</u> If η is such an equivalence class,

then $\forall f \in \eta$ ~~is~~ called a <u>representative</u> of η

~~& can write $\eta = [q; f]$~~

<u>Nota</u> given $f: \mathbb{R}^n \to \mathbb{R}^p$, $q \in \mathrm{dom} f$.

~~$[f]_q$:~~ denote the germ of f at q by $[f]_q$

~~"the germ of f at~~ q"

<u>Caution</u> : $[f]_q$ <u>not</u> a map

but an equiv class

write $[f]_q : (\mathbb{R}^n, q) \to \mathbb{R}^p$. [Every germ at q has the same value]

And we can still talk ~~about~~ $[f]_q(q)$ $\eta(q)$ (since def), $d\eta_q$ (since locally defined), hence Taylor series.

("the set of all germs at the origin of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$")

Def $\mathcal{E}_n := \{ [f]_0 \mid f: \mathbb{R}^n \rightarrowtail \mathbb{R},\ 0 \in \mathrm{dom}(f) \}$

Rmk · Ring! (by define + & × by + & mult. of representatives)

i.e. if $\underset{[f]_0}{\xi} \overset{}{,} \underset{[g]_0}{\eta} \in \mathcal{E}_n$

$U = \mathrm{dom} f \cap \mathrm{dom} g$

$\Rightarrow \xi + \eta := [f|_U + g|_U]_0$, 0

$\xi \cdot \eta := [f|_U \cdot g|_U]_0$, 1.

(Check well-definedness & ring axioms).

· Moreover, local ring ! [Prop If A: ring & $m \neq (1)$ an ideal of A s.t. $\forall x \in A \setminus m$, x is a unit (∃ inverse).

$\Rightarrow$ A is local ring w. max ideal m.]

(local ← ∃! maximal ideal)

Here $f^{-1} := \frac{1}{f}$ ← only works when $f(0) \neq 0$.

Hence (let) ~~$m_n := \{ [f]_0 \mid f(0) = 0 \} \overset{\max}{\trianglelefteq} \mathcal{E}_n$.~~

$m_n := \{ \eta \in \mathcal{E}_n \mid \eta(0) = 0 \} \overset{\max}{\triangleleft} \mathcal{E}_n$.

[since η is a germ at origin 0, $\eta(0)$ is defined]

Recall ~~Ideal generated by a set of elements is:~~

If $\eta_1, \cdots, \eta_r \in \mathcal{E}_n$,

then $\langle \eta_1, \cdots, \eta_r \rangle := \left\{ \sum_{j=1}^{r} a_j \eta_j \mid a_j \in \mathcal{E}_n \right\}$

Lazy notation

$f := [f]_0 : (\mathbb{R}^n, 0) \to \mathbb{R}$.

And write $\eta = x^2$ insted of $\eta = [(x,y) \mapsto x^2]_0$.

"generating elements maybe different"

Computation of idls eg. If $J = \langle x^2, x^2y + y^3 \rangle$ $\Rightarrow I = J$

$I = \langle x^2, y^3 \rangle$

1. x^2 gen both I & J

2. $x^2y + y^3 = (y)x^2 + y^3 \in I \Rightarrow J \subset I$

$y^3 = (x^2y + y^3) - y(x^2) \in J \Rightarrow I \subset J$.

Notice $x_1, \cdots, x_n$ (i.e. $[(x_1, \cdots, x_n) \mapsto x_i]_0$) all vanish at $\vec{0}$ $\Rightarrow x_i \in m_n$.

Great! Is $m_n \underset{\uparrow}{=} \langle x_1, \cdots, x_n \rangle$.

(**Hadamard's lem**).

Pf Let $f \in m_n$, f a rep define on $B(0, \varepsilon)$ for some $\varepsilon > 0$.

. By fundamental thm of calculus;

$$f(x_1, \cdots, x_n) = \int_0^1 \frac{d}{dt} f(tx_1, \cdots, tx_n)\, dt,$$

(Thanks for correction!)

Then from chain rule $\frac{d}{dt} f(tx_1, \cdots, tx_n) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(tx_1, \cdots, tx_n)$

Hence $f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \cdots, tx_n)\, dt}_{\in \mathcal{E}_n}$ $\in \langle x_1, \cdots, x_n \rangle$

[since $\frac{\partial}{\partial x_i}$ & $\int_0^1$ won't change the smoothness of functions] $\square$

Rmk There's a generalized version of Hadamard's lem I didn't mention in the talk which says if $I_{n+b} = m_n \cdot \mathcal{E}_b$

then $I_{n+b} = m_n = \langle x_1, \cdots, x_n \rangle$,

the pf is simply by adding b parameters behind, which is almost the same as the previous pf.

Define $m_n^r := m_n \cdot m_n^{r-1}$, for $r \geq 2$.

<u>Cor</u> m_n^r is generated by monomials of deg r in $x_1, \cdots, x_n$.

<u>Pf</u>. By induction. $r=1$: Hadama's lem.

Suppose true for $r \leq k-1$:

Let $f \in m_n^k = m_n \cdot m_n^{k-1}$

then $f = \sum_j x_j g_j$, where $g_j \in m_n^{k-1}$

$= \sum_j x_j \left(\sum_i a_i \eta_i \right)$, where $a_i \in \mathcal{E}_n$, η_i is a deg $k-1$ monomial

$= \sum_{j,i} a_i \underbrace{(x_j \eta_i)}_{\text{deg } k \text{ monomial}} \in \langle \text{monomials of deg } k \text{ in } x_1, \cdots, x_n \rangle =: M$

And in the same way can prove $M \subseteq m_n^k$

$\square$

<u>Rmk</u>
- There're $\binom{n+r-1}{r}$ deg r monomials in $x_1, \cdots, x_n$.
- Can record monomials in $\mathbb{R}[x,y]$ in a diagram by mapping

$$\{\text{monomials in } \mathbb{R}[x,y]\} \longrightarrow \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$$
$$x^a y^b \longmapsto (a,b)$$

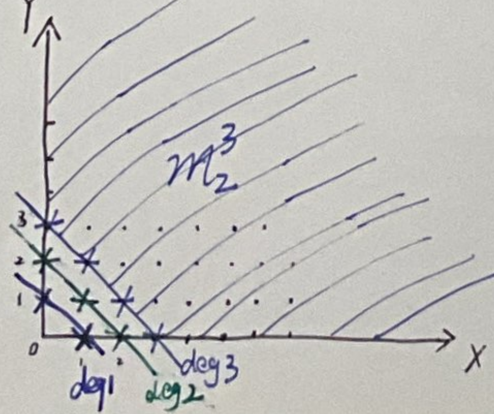
then

$$\{\text{the generators of } m_2^r\} \longmapsto \{\text{the pts in the line } X+Y=r\}$$
$$m_2^r \longmapsto \{\text{the linear combination of the pts in and above the line } X+Y=r\}$$

<u>eg</u>.

$m_2^2 = \langle y^2, xy, x^2 \rangle$ ← deg

$m_2^3 = \langle y^3, xy^2, x^2y, x^3 \rangle$



Cor $f \in \mathcal{E}_n$ is in $m_n^r \iff f$ & all its partial derivatives of order $< r$ vanish at the origin.

Pf By induction. $r=1$, $f \in m_n \iff f(0)=0$ by def of m_n.

Now suppose true for $r<k$,

"$\Rightarrow$": If $f \in m_n^k = m_n \cdot m_n^{k-1}$

then $f = gh$, for some $g \in m_n$, $h \in m_n^{k-1}$

Denote $\alpha = (\alpha_1, \cdots, \alpha_n)$

$|\alpha| = \Sigma \alpha_i$

$\alpha! = \alpha_1! \cdots \alpha_n!$

Then $\dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(0) = \dfrac{\partial^{|\alpha|}(gh)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(0)$

Notice the right handside is 0 when $|\alpha| = k-1$, since $\dfrac{\partial^{|\alpha|-l =: \beta} g}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}(0) = 0$ when $l \geq 1$ for $h \in m_n^{k-1}$

and

$(\partial^{|\alpha|} g)\, h(0) = 0$ for $g \in m_n$.

Hence $\dfrac{\cancel{\partial^{|\alpha|}}\ \partial^{|\alpha|=k-1} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(0) = 0$.

"$\Leftarrow$": If f & all its partial ~~diff~~ derivatives of order $< k$ vanish at the origin, then first by induction assumption, $f \in m_n^{k-1} = \langle \deg k-1$ monomials in $x_1, \cdots, x_n \rangle$.

Hence $f = \sum_{\substack{\alpha = (\alpha_1, \cdots, \alpha_n) \\ |\alpha| = k-1}} a_\alpha h_\alpha$, where $a_\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $h_\alpha \in \mathcal{E}_n$

Then $\alpha!\, h_\alpha(0) = \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(0) = 0$ by assumption,

Hence $h_\alpha(0) = 0$.

$\Rightarrow h_\alpha \in m_n$

$\Rightarrow f \in m_n^{k-1} m_n = m_n^k$. $\square$

Correction: In the talk, I used Taylor expansion to prove it, which is improper since $f \in \mathcal{E}_n$ ← sm. function germ maybe not-analytic, i.e. not equal to its Taylor expansion.

Def Define the <u>r-jet</u> of a function f at the origin to be its Taylor series to deg r, denoted $j^r f(0)(x_1, \cdots, x_n)$,

i.e.

$$j^r f(0)(x_1, \cdots, x_n) := \sum_{|\alpha| \le r} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(0)\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Rmk The previous corollary becomes

$f \in m_n^r \Leftrightarrow (r-1)$-jet vanish.

eg. Let $f(x,y) = e^{x+y}$,

then the 2-jet of f at the origin is

$j^2 f(0)(x,y) = 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2$,

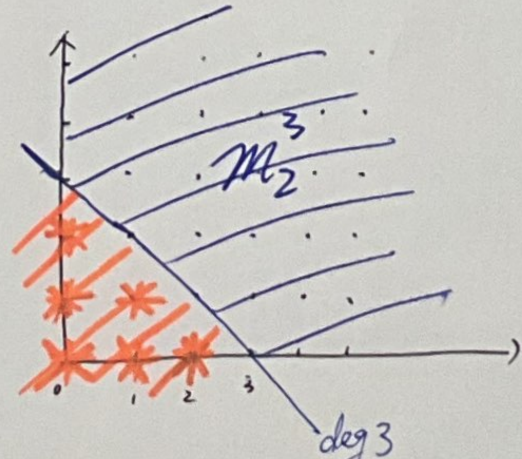
x corresponds to $(1,0)$.

Rmk • Can record more information in the previous diagram;

$\{$monoial parts of the $(r-1)$ jet$\} \longmapsto \{$the pts below the line $X+Y=r\}$

eg.

monoidal parts of the 2-jet



• can identify the k-jet of a germ $f \in \mathcal{E}_n$ with the image of f under the projection $\pi_k : \mathcal{E}_n \longrightarrow \mathcal{E}_n / m_n^{k+1}$ ring homomorphism w. kernel m_n^{k+1},
since k-jet is obtained by keeping the items up to k in the Taylor series and ignoring the rest, which can also be seen in the diagram.

Correction: In the talk, I said
$$\mathcal{E}_n \longrightarrow \mathcal{E}_n$$
$$f \longmapsto j^k f(0)(x_1, \cdots, x_n)$$
is a ring homomorphism,

But it's WRONG!

e.g. $\cancel{j^1 g(0)(x_1, \cdots, x_n) = g(0) + \sum_i \frac{\partial g}{\partial x_i}(0) x_i}$

$\cancel{j^2 g\, j^1 f(0)(x_1, \cdots, x_n) = f(0) + \sum_i \frac{\partial f}{\partial x_i}(0) x_i}$

For $f, g \in \mathcal{E}_1$,

$$j^1 g(0)(x) = g(0) + \frac{\partial g}{\partial x}(0) x$$

$$j^1 f(0)(x) = f(0) + \frac{\partial f}{\partial x}(0) x.$$

$$(j^1 f(0)(x))(j^1 g(0)(x)) = f(0)g(0) + g(0)\frac{\partial f}{\partial x}(0) x + f(0)\frac{\partial g}{\partial x}(0) x + \boxed{\frac{\partial g}{\partial x}(0)\frac{\partial f}{\partial x}(0) x^2},$$

But $j^1(fg)(0)(x) = (fg)(0) + \frac{\partial fg}{\partial x}(0) x$

$\cancel{= f(0)}$

$$= f(0)\cdot g(0) + g(0)\frac{\partial f}{\partial x}(0) x + f(0)\frac{\partial g}{\partial x}(0) x$$

$$\neq (j^1 f(0)(x))(j^1 g(0)(x))$$

Then I considered ~~But now~~ by mapping $f \longmapsto j^k f(0)(x_1, \cdots, x_n) + m_n^{k+1}$,

these Two will be the same,

i.e. $\left(j^1 f(0)(x)\right)\left(j^1 g(0)(x)\right) + m_1^{1+1} = j^1(fg)(0)(x) + m_1^{1+1}$.

But f might be some non-analytic monster (eg. $\begin{cases} e^{-\frac{1}{x}}, & x>0 \\ 0, & x\leq 0 \end{cases}$)

then it's still not a ring homomorphism.

But $\mathcal{E}_n \longrightarrow \mathcal{E}_n / m_n^{k+1}$ is always a surjective ring homomorphism for any ring and ideal,
$f \longmapsto f + m_n^{k+1}$

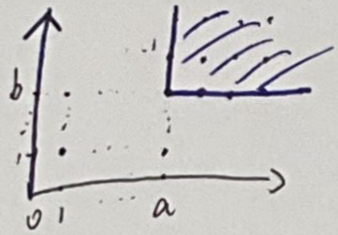
So just map f like that will be fine.

Newton diagram

The previous diagram is called Newton diagram.

a) If I is an ideal generated by a single monomial, i.e $I = \langle x^a y^b \rangle$

then can record I as



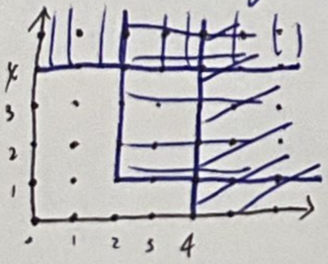
since $x^c y^d \in I \iff c \geq a \;\&\; d \geq b$,

I contains all monomials corresponding to (c,d) above and to the right of (a,b).

b) If I is generated by several monomial generates.

then shade the regions of each generator.

eg. For $I = \langle x^4, x^2y, x^3 \rangle$, the diagram is



c) If I is generated by ~~monomials~~ generators which are not monomials,

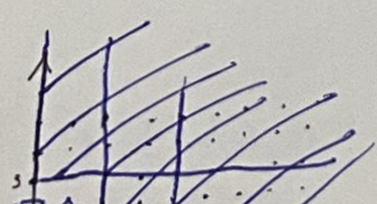
then you can choose not to draw the diagram,

but if you wanna draw, there're two steps:

Step 1. find as many generators as possible and shade in corresponding regions;

Step 2. Illustrate in some way the fact that some of the unshaded monomials are related.

eg. For $I = \langle x^2y, x^2+y^2 \rangle$



Step1 $y^3 = (x^2+y^2)\cdot y - x^2y$

$x^4 = (x^2+y^2)\cdot x^2 - (x^2y)y$

and x^2y are the generators of the monomial of the ideal

Step 2 x^2 and y^2 are marked with small square, ~~x^3 and xy~~ x^3 and xy^2 are marked with triangle illustrating the fact that x^2+y^2 and x^3+xy^2 are in the ideal but not in the part generated by monomials.

Anooter ~~to~~ depict Newton diagram
by $\mathcal{Z}$ & writing the monomials out:

eg. For $I = \langle x^4, x^2y, x^3 \rangle$

$$1$$
$$x \quad y$$
$$x^2 \quad xy \quad y^2$$
$$x^3 \quad x^2y \quad xy^2 \quad y^3$$
$$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$$
$$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$$

WANT: To show some ideal $\subset$ in another,

$\uparrow$ especially for showing I is finite codimension,

want to show $m_n^k \subseteq I$ for some k.

Need the help of Nakayama.

<u>Thm (Nakayama's lem)</u>

If R commutative ring

M an ideal s.t. $x \in M \Rightarrow 1+x$ is a unit in R

I, J ideals ~~of R~~, with I finitely generated.

important! so I only has finitely many generators.

then

$$I \subset J + MI \Rightarrow I \subset J$$

<u>Pf</u>. If $I = \langle a_1, \cdots, a_r \rangle$

Since $I \subset J + MI$,

$$\Rightarrow \quad a_i = b_i + \sum_j \lambda_{ij} a_j, \text{ for some } \lambda_{ij} \in M, \ b_i \in J$$

$$\Rightarrow \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{=\vec{a}} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}_{=\vec{b}} + \underbrace{\begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix}}_{=\Lambda} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$\Rightarrow \vec{a} = \vec{b} + \Lambda \vec{a}$$

$$\Rightarrow (\mathbb{1} - \Lambda)\vec{a} = \vec{b}$$

If $\vec{a}$ can be written as something times $\vec{b}$, then we win.

So the problem becomes if $(\mathbb{1} - \Lambda)$ is invertible.

Recall Having a formula for the inverse of a matrix A over $\mathbb{R}$ (or $\mathbb{C}$)

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

And we can have this formula for matrix over arbitrary ring R when R is commutative

and $\frac{1}{\det A} \in R$, which is the same as asking if $\det A$ is invertible in R, i.e. if $\det A$ is a unit in R.

Now back to the proof of Nakayama,

$$\det(\mathbb{1} - \Lambda) = \det\begin{pmatrix} 1-\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ -\lambda_{21} & 1-\lambda_{22} & \cdots & -\lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_{n1} & -\lambda_{n2} & \cdots & 1-\lambda_{nn} \end{pmatrix} = 1 + \underbrace{\left(\begin{matrix}\text{some multiplications and} \\ \text{additions of } \lambda_{ij}\end{matrix}\right)}_{\substack{\text{since } M \text{ is an ideal} \\ \text{and } \lambda_{ij} \in M, \\ \text{this thing is still an element of } M, \\ \text{and we can call it } \lambda}}$$

$$= 1 + \lambda$$

Then by the assumption of M, $1+\lambda$ is a unit in R.

Hence $(\mathbb{1} - \Lambda)$ is invertible,

$$\vec{a} = (\mathbb{1} - \Lambda)^{-1} \vec{b}$$

Since every ~~elem~~ generator of I can be written as some element in J,

$$I \subseteq J$$

$\square$

Rmk · Assume R to be a commutative ring since the inverse matrix formula needs it.

· If ~~A~~ m is a maximal ideal of ~~ring R~~ a local ring R then $x \in m \Rightarrow 1+x$ unit in R (if not, then $1+x = y \in m$, $1 = y - x \in m$ ↯)

· ~~Rmk~~ If $A = B + MA$, then $A = B$ (forgot to mention it in the talk)

Come back to ring $\mathcal{E}_n$, it's commutative (since $fg(0) = gf(0)$ $= f(0)g(0)$)

and maximal ideal here is $\mathfrak{m}_n$.

So now for I, J ideals of $\mathcal{E}_n$, with I finitely generated

$$I \subset J + \mathfrak{m}_n I \implies I \subseteq J$$

is what we'll use from now on.

<u>Rmk</u> If moreover $I = J + \mathfrak{m}_n I$,

then $I = J$

<u>eg.</u> Show $\mathfrak{m}_2^5 \subset \langle x^3, y^3 + x^2y^2 \rangle$.

Let $I = \mathfrak{m}_2^5$,

$J = \langle x^3, y^3 + x^2y^2 \rangle$,

want to show $I \subset J + \mathfrak{m}_2 I = J + \mathfrak{m}_2^6$

then apply Nakayama.

Denote $\alpha := x^3$,

$\beta := y^3 + x^2y^2$

then check each generator of I in turn:

$$\left.\begin{array}{l} \left.\begin{array}{l} y^5 = y^2\beta - x^2y^4 \\ xy^4 = xy\beta - x^3y^3 \\ x^2y^3 = x^2\beta - x^4y^2 \end{array}\right\} \text{all in the form of } g\beta + r, \text{ with } r \in \mathfrak{m}_2^6 \\ \left.\begin{array}{l} x^3y^2 = y^2\alpha \\ x^4y = xy\alpha \\ x^5 = x^2\alpha \end{array}\right\} \text{all} \in J \end{array}\right\} \text{all} \in J + \mathfrak{m}_2^6 = J + \mathfrak{m}_2 I$$

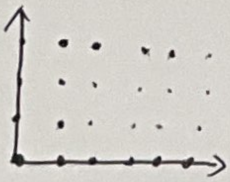
$\Rightarrow I \subset J + \mathfrak{m}_2 I$

Apply Nakayama we get $I \subset J$.

Ideals of finite codimension

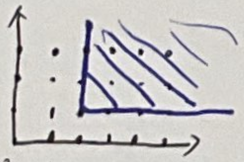
<u>Observe</u> $\mathcal{E}_n$ is a vector space but of infinite dimension,

e.g. For $\mathcal{E}_2$ all pts in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is of its basis



Also $\forall$ ideal I in $\mathcal{E}_n$ (except the zero ideal) is of infinite dim.

e.g. For $I = \langle x^2 y \rangle \triangleleft \mathcal{E}_2$, all the points in the shaded regions are the basis for I.



Hence we consider the thing between two infinite things, which may be finite.

<u>Def</u> An ideal $I \triangleleft \mathcal{E}_n$ is of <u>finite codimension</u> if $\mathcal{E}_n / I$ is a finite-dimensional vector space.

Equivalently, if $\exists$ a finite dimensional vector subspace V of $\mathcal{E}_n$ s.t. $\mathcal{E}_n = V + I$.

so $\forall$ germ $f \in \mathcal{E}_n$, f can be written as $f = g + h$, with $g \in V$, $h \in I$.

<u>Rmk</u> fin. codim is closed under addition since if $\mathcal{E}_n = \underset{\text{fin}}{V} + I$, $\mathcal{E}_n = \underset{\text{fin}}{W} + J$, then $\mathcal{E}_n = \underset{\text{fin}}{(V+W)} + I + J$.

<u>Def</u> A <u>cobasis</u> for an ideal I of finite codimension in $\mathcal{E}_n$ is a linearly independent set of elements $\{h_1, \cdots, h_r\} \subset \mathcal{E}_n$ s.t.

$$\mathcal{E}_n = \mathbb{R}\{h_1, \cdots, h_r\} \oplus I.$$

<u>Notation.</u>
- $\mathbb{F}\{v_1, v_2, \cdots, v_r\} = \{\sum_{j=1}^{r} \lambda_j v_j \mid \lambda_j \in \mathbb{F}\}$ called the <u>span</u> over $\mathbb{F}$ of $v_1, \cdots, v_r$ in V.
- the direct sum $\oplus$ means $\forall f \in \mathcal{E}_n$, $\exists!$ choice of $a_1, \cdots, a_r \in \mathbb{R}$ and $g \in I$ for which $f = \sum_j a_j h_j + g$

eg. m_n is of fin. codim in $\mathcal{E}_n$ since

$$\mathcal{E}_n = \mathbb{R} + m_n$$

This is because $\forall f \in \mathcal{E}_n$,

$$f(0) \in \mathbb{R}.$$

Now we let $\bar{f} = f - f(0)$

then $\bar{f}(0) = f(0) - f(0) = 0 \Rightarrow \bar{f}(0) \in m_n$

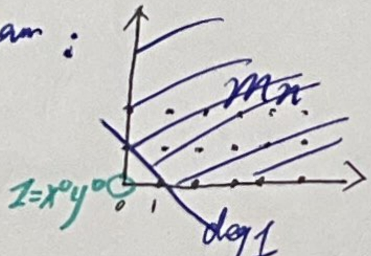
Then f can be written as

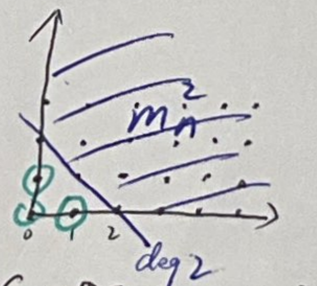
$$f = f(0) + \bar{f} \in \mathbb{R} \oplus m_n$$

<u>Rmk</u> • Moreover, $\mathcal{E}_n = \mathbb{R} \oplus m_n$ since this decomposition is unique,

so $\{1\}$ is a cobasis for $m_n \triangleleft \mathcal{E}_n$,

which can be also seen in the diagram:





$\mathcal{E}_2 = \mathbb{R}\{1, x, y\} \oplus m_n^2$

• Similarly $\mathcal{E}_n = \mathbb{R}\{1, x_1, x_2, \cdots, x_n\} \oplus m_n^2 \Rightarrow m_n^2$ is of fin. codim,

Let $V_r := \{$ all polynomials of deg $< r \}$,

then by Taylor's theorem

$$\mathcal{E}_n = V_r \oplus m_n^r \Rightarrow m_n^r \text{ is of fin. codim.}$$

Now we have m_n^r is of fin. codim, $\forall r \in \mathbb{Z}_{\geq 0} = \mathbb{N}$,

How to know if an arbitrary ideal is of fin. codim?

Prop An ideal $I \lhd \mathcal{E}_n$ is of fin. $\Longleftrightarrow \exists r \in \mathbb{N}$: s.t. $m_n^r \subset I$.

Pf "$\Leftarrow$": Suppose $m_n^r \subset I$

Define W to be $\mathbb{R}$-vector space spanned by monomials of deg $< r$.

Let $f \in \mathcal{E}_n$, $f_r :=$ the Taylor series of f to deg $r-1 = j^{r-1} f$

$\bar{f} := f - f_r$

Then $j^{r-1} \bar{f}(0) = 0$.

by ($\ast$) get $\bar{f} \in m_n^r$

Hence $f = f_r + \bar{f} \in W + m_n^r \subset W + I$

So $\mathcal{E}_n = W + I$.

Since $\dim W < \infty$, I is of fin. codim.

"$\Rightarrow$": In this direction we'll use Nakayama.

Suppose $I \lhd \mathcal{E}_n$ is of fin. codim.

$\forall r > 0$, define $I_r = I + m_n^r$, which is of fin. codim since $m_n^r \subset I + m_n^r$

(also can be got by both I and m_n^r is of fin. codim)

Then $r \nearrow$, $I_r \searrow$:

$I_1 \supset I_2 \supset \cdots \supset I_{r-1} \supset I_r \supset I_{r+1} \supset \cdots \supset I$.

Since I_r is of fin. codim,

$c_r := \dim(\mathcal{E}/I_r)$ is finite, and it follows that

$0 \leq c_1 \leq c_2 \leq \cdots \leq c_{r-1} \leq c_r \leq c_{r+1} \leq \cdots \leq c := \dim(\mathcal{E}/I)$

Since I is of fin. codim, c is finite.

Hence $\exists k$: $c_k = c_{k+1}$.

Then since $I_k \supset I_{k+1}$ &. they have the same codim,

$I_k = I_{k+1}$, i.e. $I + m_n^k = I + m_n^{k+1}$

Then $m_n^k \subset I + m_n^k = I + m_n^{k+1} = I + m_n \cdot m_n^{k}$

Applying Nakayama, get $m_n^k \subset I$ $\square$

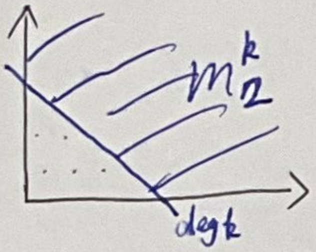
Rmk • fin. codim is closed under multiplication since if $I \supset m_n^r$, $J \supset m_n^k$ then $IJ \supset m_n^{r+k}$

• can also see the pf of the Prop by the diagram

"$\Leftarrow$": If I contains the shaded region,

then ~~have~~ only has finitely many left-lg pts outside I.

which is the cobasis of I

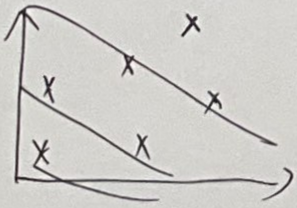


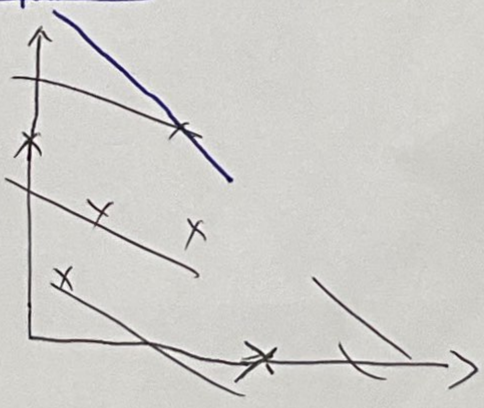
"$\Rightarrow$": If there're only fin. many pts outside I,

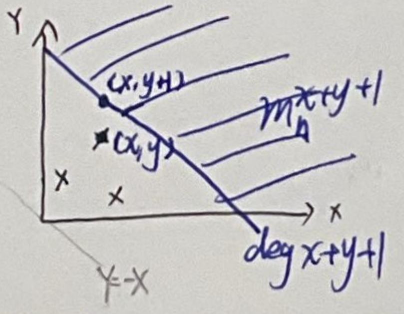
then take the pt which is the most far away from ~~$y=-x$~~, $Y=-X$

~~and~~ call it (x,y) ~~and then take x~~

then $m_2^{x+y+1} \subset I$.

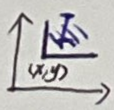


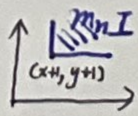




$\square$

• can also see ~~Nakayama~~ in the diagram. (monomial version) e.g.





$m_2 I$ is I move one to the right and one to the up.

Hence if $I \subset J + m_n I$, J needs to contain the pt (x,y). And once $J \ni (x,y)$, $J \supset I$.

Rmk If $J \triangleleft \mathbb{R}[x_1, \cdots x_n]$ (polynomial ring of n variable) is of fin. codim,

then let $I \triangleleft \mathcal{E}_n$ be the ideal generated by the generators of J (can do so since $\mathbb{R}[x_1,\cdots,x_n]$ is noetherian, every ideal is fin. gen.)

I is also of finite codim in $\mathcal{E}_n$, and a cobasis for J in R is also a cobasis for I in $\mathcal{E}_n$

Geometric criterion for fin. codim

For algebraically closed field, we have Nullstellensatz,

and for an analytic function germ $f \in \mathcal{E}_n$, ← sm. function germs: $\mathbb{R}^n \to \mathbb{R}$. at origin

f can be extended uniquely to a function germ from $\mathbb{C}^n \to \mathbb{C}$,

simply by replacing the x_i occurring in the Taylor series of f

with complex variables z_i.

· <u>Notation</u>: • $R :=$ the ring of germs at 0 of complex analytic functions

• If $f \in R$, and f is represented by a complex analytic function $\tilde{f} : \mathbb{C}^n \to \mathbb{C}$

then define $V_{\mathbb{C}}(f) := \{ z \in \mathbb{C}^n \mid \tilde{f}(z) = 0 \}/\sim$,

where $X \sim Y$ if for some nbh U of origin, $X \cap U = Y \cap U$

<u>Note</u>. $V_{\mathbb{C}}(f)$ is well-defined, i.e. independant of the choice of the representative of germ f.

• If $I = \langle h_1, \cdots, h_k \rangle \lhd R$,

$V_{\mathbb{C}}(I) := V_{\mathbb{C}}(h_1) \cap \cdots \cap V_{\mathbb{C}}(h_k) = \{ z \in \mathbb{C}^n \mid h_1(z) = \cdots = h_n(z) = 0 \}/\sim$

<u>Note</u> $V_{\mathbb{C}}(I)$ is also well-defined

• For subset $X \subseteq \mathbb{C}^n$,

$I_0(X) := \{ f \in R \mid V_{\mathbb{C}}(f) \supset X \}$.

<u>Thm (Analytic Nullstellensatz / Rücker's Nullstellensatz)</u>

If $I \subseteq R$, then $\sqrt{I} = I_0(V_{\mathbb{C}}(I))$, where $\sqrt{I} := \{ x \mid \exists k \in \mathbb{N} : x^k \in I \}$

We'll just use it without proof.

Cor. If $I \lhd R$, with the property that $V_{\mathbb{C}}(I) = \{0\}$, i.e. $\exists$ a nbh U of origin, s.t. $V_{\mathbb{C}}(I) \cap U = \{0\} \cap U = \{0\}$,

then I is of fin. codim in R.

Pf. $V_{\mathbb{C}}(I) = \{0\} \xrightarrow{\text{Nullstellensatz}} \sqrt{I} = I_0(V_{\mathbb{C}}(I)) = I_0(\{0\}) = \{ f \in R \mid V_{\mathbb{C}}(f) \supset \{0\} \}$
$= \{ f \in R \mid f(0) = 0 \} = m_n$

Thus $\exists n_j : x_j^{n_j} \in I$, $\forall j \in \{1, \cdots, n\}$. Let $k = \max_j n_j$, then $m_n^k \subset I$, so I is of fin. codim $\square$

Correction In the talk I wrongly said by Nullstellensatz $\sqrt{0} = m_n$.

(it's true, but not given by Nullstellensatz, true by $\bigcap_{\text{prime ideal } P} P = \sqrt{0}$, and m_n is the only maximal ideal (hence prime) in R)

And this information doesn't help to prove. (I might also say $\cancel{I \subseteq \sqrt{0}}$ in the talk, which is nonsense).

Rmk Important to take $V(I)$ in $\mathbb{C}^n$ but not in $\mathbb{R}^n$,

or have e.g. $I = \langle x^2 + y^2 \rangle \subset m_2$ ~~but~~ not have fin. codim.

For ~~$V_{\mathbb{R}}$~~ although $V_{\mathbb{R}}(I) = \{0\}$,

$V_{\mathbb{C}}(I) = \{ (x,y) \mid x = \pm iy \}$ consists of two complex lines,

so the origin is not isolated.

Still unknow for me By the corollary of Nullstellensatz,

we can have

~~If~~ $V_{\mathbb{C}}(I) = \{0\} \Rightarrow I$ is of fin. codim in R

cmplx analytic function germ,

but I don't know ~~how to prove~~ if it implies that I is of ~~co~~ fin. codim in $\mathcal{E}_n$,

But let's assume it's so.

Thm (Geometric criterion)

Let $I = \langle h_1, \cdots, h_k \rangle \subset m_n$ be an ideal generated by finitely many analytic germs

Then

I has fin. codim in $\mathcal{E}_n \iff V_{\mathbb{C}}(I) = \{0\}$.

Pf. "$\Leftarrow$": assumed to be true.

"$\Rightarrow$": ~~$V_{\mathbb{C}}(I) \subset$~~

I has fin codim $\iff \exists r \in \mathbb{N} : m_n^r \subset I$

then $\{0\} = V_{\mathbb{C}}(m_n) \subseteq V_{\mathbb{C}}(I) \subseteq V_{\mathbb{C}}(m_n^r) = \{0\}$

$\square$