Def 1 A ~~smooth~~ family of maps $f_u : \mathbb{R}^n \to \mathbb{R}^p$ $(u \in \mathbb{R}^q)$ is a <u>smooth family</u> if $F : \mathbb{R}^q \times \mathbb{R}^n \to \mathbb{R}^p$, $F(u, x) = f_u(x)$ is smooth.

Similarly for smooth family of germs

~~**Def 2** A vector field on $U \subset \mathbb{R}^n$ is a vector $v(x) \in \mathbb{R}^n$ at each point $x \in U$. smooth v.f. if v depends smoothly on x.~~

The <u>germ of a v.f.</u> at $q \in \mathbb{R}^n$ is the equiv. class $[v]_q$ of v.f.s that coincide w/ v in some nbhd of q.

<u>θ_n</u> := set of germs at 0 of smooth v.f. on $\mathbb{R}^n$.

Def 3 $f : \mathbb{R}^n \to \mathbb{R}^p$ smooth map, v vector field on dom(f).

Then $tf(v)(x) = df_x(v(x)) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x) \cdot v_j(x)$.

tf is the tangent map of f.

Ex $f(x,y) = x^2 + y^3 \in \mathcal{E}_2$, $v(x,y) = \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} \in \theta_2$

$tf(v)(x,y) = (2x \quad 3y^2)\begin{pmatrix} y^2 \\ x^2 \end{pmatrix} = 2xy^2 + 3x^2y^2.$

$p = 1 \Rightarrow tf(v) \in \mathcal{E}_n$; $tf : \theta_n \to \mathcal{E}_n$.

vary $v \Rightarrow tf(\theta_n) = Jf$.

Def 4. A 1-parameter family of germs $f_s : (\mathbb{R}^n, 0) \to \mathbb{R}$ ($s \in I$ an interval w/ $0 \in I$) is said to be a <u>trivial family</u> if $(\exists)$ smooth family of diffeo. germs ϕ_s, $s \in I$, s.t. $\phi_0 = id$ & $f_s \circ \phi_s = f_0$.

$\Rightarrow$ must have $\phi_s(0) = 0$ bc. all germs at 0

Thm 5 (Thom-Levine principle) Let $f_s \in \mathcal{E}_n$ be a smooth family of germs ($s \in I$). Then (f_s) is a trivial family $\Leftrightarrow$ $\dot{f}_s \in \mathfrak{m}_n Jf_s$ smoothly in s.

"$\frac{d}{ds} f_s$."

smoothly in s: $(\exists)$ smooth functions $u_j(x, s)$ w/ $u_j(0, s) = 0$ s.t. $\dot{f}_s(x) = \sum_{j=1}^{n} u_j(x, s) \cdot \frac{\partial f_s}{\partial x_j}(x)$.

$\downarrow$ germs in x, def'd for all $s \in I$.

Lemma A $U \subseteq \mathbb{R}^n$ open set, $f_s : U \to \mathbb{R}^n$ smooth family of smooth maps; $\phi_s : U \to U$ another smooth family of smooth maps $\Rightarrow$

$$\frac{d}{ds}(f_s \circ \phi_s(x)) = \dot{f}_s(y) + (df_s)_y(v_s(y)),$$

where $y = \phi_s(x)$, $v_s(y) = \frac{d}{ds}\phi_s(x)$.

Lemma B v_t is a smooth time-dependent v.f. def^d on an open subset U of $\mathbb{R}^n$, ~~[illegible]~~; $x_0 \in U$ & $v_t(x_0) = 0$, $(\forall) t$. $t \in I$ compact interval $\ni 0$

Then $(\exists)$ $V \subset U$ nbhd of x_0 and $\phi : V \times I \to U$ s.t.

$$\begin{cases} \frac{\partial}{\partial t} \phi(x,t) = v_t(\phi(x,t)) \\ \phi(x,0) = x \end{cases}$$

$\phi_t(x) = \phi(x,t)$ is called the flow of v, ~~it is smooth,~~ ~~each ϕ_t a diffeo for small $t \in (-\varepsilon, \varepsilon)$~~. ϕ_t are diffeo.

Proof of Thom-Levine

"$\Rightarrow$" Suppose $(f_s)_s$ is trivial, $f_s \circ \phi_s = f_0$, ϕ_s smooth family of diffeo. Take $\frac{d}{ds}$ $\Rightarrow$

$\Rightarrow \dot{f}_s(y) + (tf_s)(v_s)(y) = 0$, $\quad y = \phi_s(x)$, $v_s(y) = \frac{d}{ds}\phi_s(x)$.

$\phi_s(0) = 0 \overset{\frac{d}{ds}}{\Rightarrow} v_s(0) = 0$.

If we write $v_s(x) = ((v_s)_1(x), (v_s)_2(x), \dots, (v_s)_n(x))$, then $\dot{f}_s = -\sum_{j=1}^{n} (v_s)_j \cdot \frac{\partial f_s}{\partial x_j} \in m_n Jf_s$.

"$\Leftarrow$" Write $\dot{f}_s(x) = \sum_{j=1}^{n} u_j(x,s) \cdot \frac{\partial f_s}{\partial x_j}(x)$. (take representatives for germs)

~~Take the~~ Let $u(x,t) = (u_1(x,t), u_2(x,t), \dots, u_n(x,t))$.

$\phi_t(x)$ flow of $-u(x,t)$ (every t can be put in a compact interval containing 0)

$\frac{d}{ds}(f_s \circ \phi_s)(x) = \dot{f}_s(y) + (df_s)_y(-u_s(y)) = 0$.

$\Rightarrow (f_s \circ \phi_s)(x)$ ct in s $\Rightarrow$

$\Rightarrow f_s \circ \phi_s = f_0 \circ \phi_0 = f_0$. ■

Def 6 The right tangent space to $f \in \mathcal{E}_n^m$ is the ideal

$$TR \cdot f = tf(m_n \Theta_n) = m_n Jf.$$ ~~$tf(\mathcal{E}_n)$~~

Prop 7 $f \in \mathcal{E}_n$, s.t. $m_n^k \subset TR \cdot f$ i.e. $m_n^k \subset m_n Jf$. Let $h \in m_n^{k+1}$ and $g = f + h$. Then

$$TR \cdot f = TR \cdot g.$$

So $TR \cdot f$ depends only on the k-jet of f.

Proof Let $I = m_n^k$; $h \in m_n I \Rightarrow \frac{\partial h}{\partial x_i} \in I \Rightarrow$

$\Rightarrow \frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i} \in I \Rightarrow Jf \subseteq Jg + I$

(1) $\Rightarrow m_n Jf \subseteq m_n Jg + m_n I$ and similarly

(2) $m_n Jg \subseteq m_n Jf + m_n I$.

(2) $\Rightarrow m_n Jg \subseteq m_n Jf + m_n^2 Jf = m_n Jf$.

(1) $\Rightarrow m_n Jf \subseteq m_n Jg + m_n^2 Jf \overset{\text{Nakayama}}{\Longrightarrow} m_n Jf \subseteq m_n Jg$. ■

Finite determinacy

Def 8 A function germ $f \in \mathcal{E}_n$ is said to be *k-determined* if, whenever $g \in \mathcal{E}_n$ satisfies $j^k g(0) = j^k f(0)$, we have $g \sim_{\mathcal{R}} f$.
We say f is *finitely determined* if it is k-det^d for some $k \in \mathbb{N}^*$.

Example 10 (Olivia's question) Let $f, g \in \mathcal{E}_1$, $f(x) = ax^k + h(x)$, $k \geq 1$, $a \neq 0$, $h \in \underline{m}_1^{k+1}$, $g(x) = ax^k$.

Then $Jg = \langle x^{k-1} \rangle = \underline{m}_1^{k-1}$ $\underset{\text{finite determinacy}}{\Longrightarrow}$ $\left.\begin{array}{l} g \text{ is k-det}^d \\ j^k g(0) = j^k f(0) \end{array}\right\} \Rightarrow$

$\Rightarrow f \sim_{\mathcal{R}} g$.

Thm 9 (finite determinacy)
If $f \in \mathcal{E}_n$ satisfies $\underline{m}_n^{k+1} \subseteq \underline{m}_n^2 Jf$, then f is k-det^d.

$\underline{m}_n^{k-1} \subseteq Jf \Rightarrow \underline{m}_n^k \subseteq \underline{m}_n Jf \Rightarrow \underline{m}_n^{k+1} \subseteq \underline{m}_n^2 Jf$

possible conditions.

Example 11 (i) $f = x^3 + y^3 \Rightarrow Jf = \langle x^2, y^2 \rangle \Rightarrow$
$\Rightarrow \underline{m}_2 Jf = \langle x, y \rangle \langle x^2, y^2 \rangle =$
$= \underline{m}_2^3 \Rightarrow f$ is 3-det^d.

(ii) $f = x^4 + y^4 \Rightarrow Jf = \langle x^3, y^3 \rangle \Rightarrow$
$\Rightarrow \underline{m}_2 Jf = \langle x^4, x^3 y, xy^3, y^4 \rangle$
$\Rightarrow \underline{m}_2^2 Jf = \langle x^5, x^4 y, x^3 y^2, x^2 y^3, xy^4, y^5 \rangle$

(iii) $f = x^5 + y^5 \Rightarrow Jf = \langle x^4, y^4 \rangle \Rightarrow$
$\Rightarrow \underline{m}_2 Jf = \langle x^5, x^4 y, xy^4, y^5 \rangle \Rightarrow$
$\Rightarrow \underline{m}_2^2 Jf = \langle x^6, x^5 y, x^4 y^2, x^2 y^4, xy^5, y^6 \rangle$
$\Rightarrow \underline{m}_2^2 Jf \supseteq \underline{m}_2^7$ but $\underline{m}_2^2 Jf \neq \underline{m}_2^6$.
$\Rightarrow f$ is 6-det^d

If $g = x^3 y^3$ and $f_s(x,y) = f(x,y) + s\, g(x,y)$, then
$f_s(x,y) - g(x,y) = x^3 y^3 \notin Jf \Rightarrow f_s \notin \underline{m}_2 Jf \Rightarrow$
$\left.\begin{array}{l} \Rightarrow (f_s) \text{ is not a trivial family} \\ \text{Yet all } f_s \text{ have the same 5-jet} \end{array}\right\} \Rightarrow$

$\overset{12}{\Rightarrow} f$ is NOT 5-det^d.

Corollary 12 (Morse's lemma)
$f \in \mathcal{E}_n$ has a nondegenerate critical point at $0 \Rightarrow$
$Jf = \underline{m}_n \Rightarrow f$ is 2-det^d.

$\xRightarrow{\text{previous talk}}$ $f(x) = \pm x_1^2 \pm \ldots \pm x_n^2 \Rightarrow 0$ is nondeg. critical point.

Prop 13 (partial converse)
$f\in \mathcal{E}_m$ is k-det^d. Then $\underline{m}_m^{k+1} \subseteq TR\cdot f = \underline{m}_m Jf$.

~~Combining all of these, we get~~
~~$\underline{m}_m^{k+1} \subseteq \underline{m}_m^2 Jf \Rightarrow \underline{m}_m^{k+1} \subseteq \underline{m}_m Jf$~~

Corollary 14 A germ $f\in \mathcal{E}_m$ is fin. det^d $\Leftrightarrow$ it is of finite codimension.

Proof of finite determinacy

- $\mathcal{E}_{m,I}$ = ring of germs of smooth functions along $\{0\}\times I$, $I=[0,1]$.

$\underline{m}_{m,I} = \{ g\in \mathcal{E}_{m,I} \mid g(0,0,\dots,0,s)=0, (\forall) s\in[0,1]\} =$
$= \langle x_1,\dots,x_m\rangle$ by Hadamard's lemma.

- Take $f\in \mathcal{E}_m$ s.t. $\underline{m}_m^{k+1} \subseteq \underline{m}_m^2 Jf$. We want to prove f is k-det^d.

So take $h\in \underline{m}_m^{k+1}$ and let us prove $f+h \sim_R f$ i.e. let us find a diffeo. ϕ around 0 s.t. $(f+h)\circ\phi = f$.

Define $f_s = f+sh$, $s\in[0,1]$. For each s, we want a diffeo. germ $\phi_s : (\mathbb{R}^m,0)\to(\mathbb{R}^m,0)$ s.t. $f_s\circ\phi_s = f$. We differentiate w.r.t. s to get the <u>infinitesimal homotopy equation</u>

$$\dot{f}_s(y) + (df_s)_y(v_s(y)) = 0,$$

where $y=\phi_s(x)$ & $v_s(y) = \frac{d}{ds}\phi_s(x)$. The equation rereads $(df_s)_y(v_s(y)) = -h(y) \Leftrightarrow$

$$\Leftrightarrow v_{1,s}(y)\cdot \frac{\partial f_s}{\partial y_1} + \dots + v_{m,s}(y)\cdot\frac{\partial f_s}{\partial y_m} = -h(y). \quad (*)$$

$h\in \underline{m}_m^{k+1} \subseteq \underline{m}_m^2 Jf \Rightarrow h\in \underline{m}_m^2 Jf_s$ (lemma below)

$\Rightarrow (\exists)\ v_{i,s}\in \underline{m}_{m,I}^2$ s.t. $(*)$ holds.

For fixed $s\in[0,1]=I$, define
$$v_s(y) = (v_{1,s}(y),\dots,v_{m,s}(y)).$$

v_s has a flow ϕ_s s.t. $\phi_s(0)=0$, $(\forall) s\in[0,1]$, all ϕ_s defined on some neighbourhood of 0, and $\phi_0 \equiv id$.

$$\frac{d}{ds}(f_s\circ\phi_s) = \dot{f}_s(\phi_s(x)) + (df_s)_{\phi_s(x)}(v_s(\phi_s(x)))$$
$$\overset{(*)}{=} + h(\phi_s(x)) + (-h(\phi_s(x))) = 0$$

$\Rightarrow f_s\circ\phi_s = f_0\circ\phi_0 = f_0 = f$ ✓

$\Rightarrow (f+h)\circ\phi_1 = f$. ∎

Lemma below $f, h \in \mathcal{E}_n$ s.t. $\mathfrak{m}_n^{k+1} \subseteq \mathfrak{m}_n^2 Jf$ and $h \in \mathfrak{m}_n^{k+1}$.
Then, for $f_s = f + sh$, $\mathfrak{m}_{n,I}^{k+1} \subseteq \mathfrak{m}_{n,I}^2 Jf_s$.

Proof We prove $\mathfrak{m}_{n,I}^{k+1} \subseteq \mathfrak{m}_{n,I}^2 Jf_s + \mathfrak{m}_{n,I}^{k+2}$.
relation clear for $s = 0$; for $s \in [0,1]$,
hypothesis $\Rightarrow \mathfrak{m}_{n,I}^{k+1} \subseteq \mathfrak{m}_{n,I}^2 Jf \subseteq$

$$\subseteq \mathfrak{m}_{n,I}^2 (Jf_s + Jh) =$$

$$= \mathfrak{m}_{n,I}^2 Jf_s + \mathfrak{m}_{n,I}^2 Jh \subseteq$$

$\{h \in \mathfrak{m}_n^{k+1} \Rightarrow Jh \subseteq \mathfrak{m}_n^k\}$ $\subseteq \mathfrak{m}_{n,I}^2 Jf_s + \mathfrak{m}_{n,I}^{k+2} \xrightarrow{\text{Nakayama}}$

$$\Rightarrow \mathfrak{m}_{n,I}^{k+1} \subseteq \mathfrak{m}_{n,I}^2 Jf_s. \quad \blacksquare$$

Theorem 15 An analytic germ $f \in \mathcal{E}_n$ is finitely det^d iff the origin is an isolated critical point for $f^{\mathbb{C}}$.

- proof w/ Nullstellensatz
- $f(x,y) = (x^2+y^2)^2$; 0 is an isolated critical point in $\mathbb{R}^2$, but not in $\mathbb{C}^2 \Rightarrow f$ not fin det^d.

Two geometric interpretations of the codim of a critical point; let $\mu(f) = 1 + \text{codim}(f)$, the Milnor number.

- f complex analytic germ of fin. codim., one can find a small analytic perturbation f_ε of f w/ only nondegenerate critical points
  $\#$ of critical points is $\mu(f)$.
- (+) $\delta \in \mathbb{C}$ suff. small, the Milnor fiber $f^{-1}(\delta) \cap U$ (U nbhd. of 0 in $\mathbb{C}^n$) is a submfld of real dim. $2n-2$; this submfld. is homotopic to a wedge sum of $\mu(f)$ spheres of dim. $n-1$.