Classification of the elementary catastrophes

classify critical points up to right equivalence

goal: find 1 germ per equivalence class. This is impossible: infinitely many in infinite families

restrict to germs of codimension at most 4

$f: (\mathbb{R}^n, 0) \to \mathbb{R}$ function germ with critical point at origin. suppose $f(0)=0$ so $f \in m_n^2$. $H_f = d^2 f_0$ Hessian at the origin. H_f non-degenerate: $f \sim \sum \pm x_i^2$ by Morse Lemma. Classification for non degenerate critical points

H_f degenerate. Def: corank of f at 0 is $\dim(\ker H_f)$. $\ell := \operatorname{corank}(f)$. By splitting Lemma there are coordinates s.t. $f(x,y) = h(x) + \sum_{i=1}^{n-\ell} \pm y_i^2 \quad (x \in \mathbb{R}^\ell, y \in \mathbb{R}^{n-\ell})$

To classify germs on $\mathbb{R}^n$ of corank ℓ we only need to classify functions of ℓ variables

<u>Prop</u> (see also: 3.5 in the May 3rd talk on right equivalence): Let $f \in \mathcal{E}_{m+k}$ with $f(x,u) = \sum_{i=1}^{m} \pm x_i^2 + h(u)$ $(x \in \mathbb{R}^m, u \in \mathbb{R}^k)$. Then $\operatorname{codim}(f) = \operatorname{codim}(h)$.

The proof is based on the following lemma:

Lemma: Let R, S be rings, $J \lhd S$ an ideal. Suppose $\phi: R \to S$ is a surjective homomorphism, $I = \phi^{-1}(J)$. Then $I \lhd R$ and ϕ induces an isomorphism $\bar{\phi}: R/I \to S/J$ defined by $\bar{\phi}(r+I) = \underbrace{\phi(r)+J}_{\in J}$.

Proof of Lemma: Compose ϕ with the natural homomorphism $\pi: S \to S/J$, $s \mapsto s+J$.

$\varphi := \pi \circ \phi: R \to S/J$, $r \mapsto \phi(r)+J$ is a ring homomorphism. From the first isomorphism theorem for rings we know that $\ker\varphi$ is an ideal, and φ induces an isomorphism $\bar{\varphi}: R/\ker\varphi \xrightarrow{\sim} \operatorname{im}\varphi$. We have $\ker\varphi = \varphi^{-1}(J) = \phi^{-1}(J) = I$, and $\operatorname{im}\varphi = S/J$ because ϕ is surjective. So $\bar{\phi} = \bar{\varphi}: R/I \to S/J$, $r+I \mapsto \phi(r)+J$ $\square$

Proof of Prop: The jacobian ideal $J_f = \langle x_1, \dots, x_m, \frac{\partial h}{\partial u_1}, \dots, \frac{\partial h}{\partial u_k} \rangle$. Let $\phi: \mathcal{E}_{m+k} \to \mathcal{E}_k$ be the homomorphism obtained by setting $x=0$: $\phi(g)(u) = g(0,u)$. ϕ is surjective, since for any $h \in \mathcal{E}_k$, $H \in \mathcal{E}_{m+k}$ with $H(x,u) := h(u)$ satisfies $\phi(H) = h$. Let $g \in \mathcal{E}_{m+k}$, then $g \in \mathcal{J}_f$ iff $\phi(g) \in J_h$:

"$\Rightarrow$" $g \in \mathcal{J}_f$. Then we can write $g(x,u) = \sum_{i=1}^{m} \varphi_i(x,u) x_i + \sum_{j=1}^{k} \psi_j(x,u) \frac{\partial h}{\partial u_j}(u)$ for some $\varphi_i, \psi_j \in \mathcal{E}_{m+k}$. Therefore $\phi(g)(u) = g(0,u) = \sum_{j=1}^{k} \psi_j(0,u) \frac{\partial h}{\partial u_j}(u)$, so $\phi(g) \in J_h$.

"$\Leftarrow$" $\phi(g) \in J_h$. Then there exists a $\tilde{g} \in \mathcal{E}_{m+k}$ s.t. $\tilde{g}(0,u) = 0$ and $g(x,u) = \tilde{g}(x,u) + \phi(g)(u)$.

$\tilde{g} \in \{\varphi \in \mathcal{E}_{m+k} \mid \varphi(0,u) = 0\}$. From Hadamard's Lemma we know that $\{\varphi \in \mathcal{E}_{m+k} \mid \varphi(0,u) = 0\} = \langle x_1, \dots, x_m \rangle$, so $\tilde{g} \in \langle x_1, \dots, x_m \rangle$ and $g = \tilde{g} + \phi(g) \in \mathcal{J}_f$

Now we can apply our Lemma to $\phi: \mathcal{E}_{m+k} \to \mathcal{E}_k$, $J_h \lhd \mathcal{E}_k$, $\mathcal{J}_f = \phi^{-1}(J_h)$, which tells us that there is an isomorphism $\bar{\phi}: \mathcal{E}_{m+k}/\mathcal{J}_f \to \mathcal{E}_k/J_h$, and in particular $\operatorname{codim}(f) = \dim(\mathcal{E}_{m+k}/\mathcal{J}_f) = \dim(\mathcal{E}_k/J_h) = \operatorname{codim}(h)$.

Lemma 6.1: $\operatorname{corank}(f) = \ell \Rightarrow \operatorname{codim}(f) \geq \frac{1}{2}\ell(\ell+1)$

> Rem: in particular - $\operatorname{corank}(f) > 2 \Rightarrow \operatorname{codim}(f) \geq 6$, so to classify all germs up to codim 4 we only need to consider germs of corank 1 + 2

Proof: Show that $m_\ell/(J_h + m_\ell^3)$ has dim at least $\frac{1}{2}\ell(\ell+1)$. J_h has ℓ generators, all in m_ℓ^2, so $J_h + m_\ell^3 \cong \mathbb{R}^{\tilde{\ell}} \oplus m_\ell^3$ for some $\tilde{\ell} \leq \ell$. Now m_ℓ/m_ℓ^3 has dim $\ell + \frac{1}{2}\ell(\ell+1)$, and taking the quotient with J_h reduces the dimension by at most $\tilde{\ell}$, so $\dim(m_\ell/(J_h + m_\ell^3)) = \ell + \frac{1}{2}\ell(\ell+1) - \tilde{\ell} \geq \frac{1}{2}\ell(\ell+1)$

<u>Classification of corank 1 singularities</u>

For functions of one variable one can give a complete classification of germs of finite codimension, so here we do not restrict the codimension. This is not possible for 2 or more variables.

If the singularity is of corank 1, the splitting lemma reduces the problem to one variable, the other variables appearing as squares as seen above: $f(x,y) = h(x) + \sum_{i=1}^{n-1} \pm y_i^2 \quad (x, y_i \in \mathbb{R})$

Theorem 6.2: Let $f \in m_1^2$ be of finite codimension. Then $\exists\, k>1$ s.t. $f \sim_R \pm x^k$

Proof: f is of finite codimension => Taylor series of f starts at some order, say $f(x) = ax^k + x^{k+1} h(x)$ where h is a smooth function and $a \neq 0$. We know by the finite determinacy th. that such an f is equivalent to ax^k. By rescaling x by $\phi(x) = |a|^{-\frac{1}{k}} x$ we obtain $f \circ \phi = \mathrm{sign}(a) x^k$, so $f \sim_R \mathrm{sign}(a) x^k$ $\square$

Rem: If k is odd, $+x^k \sim -x^k$ by $x \mapsto -x$, if k is even $x^k \not\sim -x^k$ => any finite codimension germ in one variable is right equivalent to one of $\begin{matrix} x^2 \\ -x^2 \end{matrix}\ x^3\ \begin{matrix} x^4 \\ -x^4 \end{matrix}\ x^5 \dots$

A germ equivalent to $\pm x^{k+1}$ is called a critical point of type $A_k^\pm$. For even k $A_k^+ = A_k^- =: A_k$, for odd k $A_k^+ \neq A_k^-$. Critical points of type $A_k^\pm$ are of codimension $k-1$

type :	$A_1^\pm$	A_2	$A_3^\pm$	A_4
normal form :	$\pm x^2$	x^3	$\pm x^4$	x^5
codimension :	0	1	2	3

Classification of corank 2 critical points

Again: apply splitting lemma to reduce the problem to 2 variables, consider classification of critical points of the form $f(x,y)$ satisfying $d^2 f(0) = 0$, i.e. $f \in m_2^3$

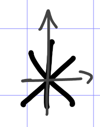
The 2-jet of $f \in m_2^3$ is zero, so the 3-jet must be of the form $j^3 f = ax^3 + bx^2 y + cxy^2 + dy^3$ $(a,b,c,d \in \mathbb{R})$

This is a "binary cubic form" (a homogeneous cubic polynomial in 2 variables).

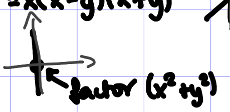
Classification of binary cubic forms: $C = ax^3 + bx^2y + cxy^2 + dy^3$, $a,b,c,d \in \mathbb{R}$

first: assume $a \neq 0$. Factorize C over $\mathbb{C}$ as $C = a(x-\alpha y)(x-\beta y)(x-\gamma y)$ (for example $\frac{d}{a} = -\alpha\beta\gamma$)

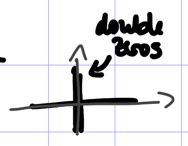
Then • (i) roots α, β, γ are real and distinct "elliptic cubic". After linear change in coordinates: $C = x^3 - xy^2 = x(x-y)(x+y)$



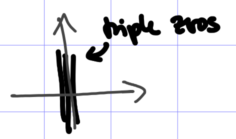
• (ii) roots α, β, γ are distinct but not all real. wlog $\alpha \in \mathbb{R}$, $\beta = \bar{\gamma}$ "hyperbolic cubic". Linear change in coordinates: $C = x^3 + xy^2 = x(x^2+y^2) = x(x+iy)(x-iy)$



• (iii) double root $\alpha = \beta \neq \gamma$ "parabolic cubic". $C = a(x-\alpha y)^2 (x-\gamma y)$. Replacing $(x-\alpha y)$ with x and $(x-\gamma y)$ with y and rescaling with $|a|^{\frac{1}{3}}$ leads to $C = x^2 y$



• (iv) three equal roots $\alpha = \beta = \gamma$ "symbolic cubic" or "perfect cube". $C = a(x-\alpha y)^3$. Replacing (x,y) by $(x-\alpha y, y)$ and rescaling by $|a|^{\frac{1}{3}}$ gives $C = x^3$



Now if $a = 0$, assume $d \neq 0$ and repeat the argument above with x and y exchanged.

If $a = d = 0$: $C = bx^2y + cxy^2 = xy(bx+cy)$

If $b = c = 0$: (v) zero form $C = 0$

If only one of b,c is non-zero we have a parabolic form (x^2y or xy^2)

If both $b \neq 0, c \neq 0$, C has three distinct real factors and must therefore be an elliptic cubic.

These are all possible cases.

* zero set picture* (double / triple lines: double/triple roots, dot at origin: factor (x^2+y^2))

Types of 3-jet of f: elliptic, hyperbolic, parabolic, symbolic

Elliptic umbilic - after linear change in coordinates $f(x,y) = x^2 y - y^3 + h(x,y)$, $h \in m_2^4$. Claim: f and $j^3 f$ are equivalent. $f_0 := x^2y - y^3$. Then $J_{f_0} = \langle xy, x^2 - 3y^2 \rangle$, so $m_2 J_{f_0} = \langle x^2y, xy^2, x^3 - 3xy^2, x^2y - 3y^3 \rangle = m_2^3$. Therefore f is 3-determined, so f is equivalent to f_0. Denote elliptic umbilic by D_4^-

Hyperbolic umbilic - after a linear change in coordinates $f(x,y) = x^2y + y^3 + h(x,y)$. The same calculation as above shows that it is equivalent to its 3-jet $f_0 = x^2y + y^3$. Denote hyperbolic umbilic by D_4^+

Parabolic umbilic - after a linear change in coordinates $f(x,y) = x^2y + h(x,y)$. The 3-jet x^2y is not 3-determined (not even finitely determined). We therefore look at the 4-jet $j^4 f =: \tilde{f} = x^2y + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$ and assume $e \neq 0$

Prop: The germ $\hat{f}$ (see above) with $e \neq 0$ is right equivalent to $f_0 := x^2y \pm y^4$

Proof: (i) (Nakayama: $I, J \trianglelefteq \mathcal{E}_n$, I, fin gen then $I \subseteq J + \mathfrak{m}_n I \Rightarrow I \subseteq J$. $I = \mathfrak{m}_2^4$, $J = \mathfrak{m}_2 J_{\hat{f}}$)

First, show that $\mathfrak{m}_2^4 \subseteq \mathfrak{m}_2 J_{\hat{f}} + \mathfrak{m}_2^5$:

$$\mathfrak{m}_2 J_{\hat{f}} = \mathfrak{m}_2 \left\langle \frac{\partial \hat{f}}{\partial x}, \frac{\partial \hat{f}}{\partial y} \right\rangle = \left\langle x\frac{\partial \hat{f}}{\partial x}, y\frac{\partial \hat{f}}{\partial x}, x\frac{\partial \hat{f}}{\partial y}, y\frac{\partial \hat{f}}{\partial y} \right\rangle$$

$$= \langle \underbrace{2x^2y + 4ax^4 + 3bx^3y + 2cx^2y^2 + 3dxy^3}_{=:\alpha}, \underbrace{x^3 + bx^4 + 2cx^3y + 3dx^2y^2 + 4exy^3}_{=:\beta}, \underbrace{2xy^2 + 4ax^3y + 3bx^2y^2 + 2cxy^3 + 3dy^4}_{=:\gamma}, \underbrace{x^2y + bx^3y + 2cx^2y^2 + 3dxy^3 + 4ey^4}_{=:\delta} \rangle$$

Check the generators of $\mathfrak{m}_2^4$: $x^4 = x\beta - (\underbrace{bx^5 + 2cx^4y + 3dx^3y^2 + 4ex^2y^3}_{\in \mathfrak{m}_2^5})$, $x^3y = x\delta - (\underbrace{bx^4y + 2cx^3y^2 + 3dx^2y^3 + 4exy^4}_{\in \mathfrak{m}_2^5})$

$x^2y^2 = y\delta - (\underbrace{bx^3y^2 + 2cx^2y^3 + 3dxy^4 + 4ey^5}_{\in \mathfrak{m}_2^5})$

$x^3y = y\beta - (\underbrace{bx^4y + 2cx^3y^2 + 3dx^2y^3 + 4exy^4}_{\in \mathfrak{m}_2^5})$, $y^4 =$

$$y^4 = \frac{1}{8e}\Big(-(cy+1)\alpha + (4ax+by)\beta - \frac{3}{2}dy\gamma + 2\delta - \big(\underbrace{4abx^5 + (4ac+b)x^4y) + (6ad-bc)x^3y + (16ae - \tfrac{3}{2}bd - 2c^2)x^2y^3 + (4be-6cd)xy^4 - \tfrac{9}{2}d^2y^5}_{\in \mathfrak{m}_2^5}\big)\Big)$$

(We assumed $e \neq 0$)

By Nakayama's Lemma, we know that $\mathfrak{m}_2^4 \subseteq \mathfrak{m}_2 J_{\hat{f}} + \mathfrak{m}_2^5 \Rightarrow \mathfrak{m}_2^4 \subseteq \mathfrak{m}_2 J_{\hat{f}}$. It therefore follows from the finite determinacy theorem that $\hat{f}$ is 4-determined.

(ii) We change the coordinates s.t. $x = X + \alpha$, $y = Y + \beta$ where $\alpha, \beta \in \mathfrak{m}_2^2$. Using our new coordinates and disregarding any terms of degree at least 5 (we can do this since because of the 4-determinacy we just proved), we get the following expression: $X^2Y + 2XY\alpha + X^2\beta + aX^4 + bX^3Y + cX^2Y^2 + dXY^3 + eY^4$

Choosing α and β to be $\alpha = -\frac{1}{2}(bX^2 + dY^2)$ and $\beta = -(aX^2 + cY^2)$ leaves us with $X^2Y + eY^4$.

(iii) By (i) and (ii) we have the equivalence of $\hat{f}$ and $x^2y + ey^4 = x^2y \pm |e|y^4$. By rescaling y ($y \mapsto \frac{y}{|e|^{\frac{1}{4}}}$ (we assumed that $e \neq 0$)) we get right equivalence to $\frac{1}{|e|^{\frac{1}{4}}} \cdot x^2y \pm y^4$, and by rescaling x ($x \mapsto |e|^{\frac{1}{8}}x$) we obtain the equivalence $\hat{f} \sim_R x^2y \pm y^4$.

Symbolic cubic - it can be shown that the codimension of this case is at least 5. The first singularity with this 3-jet is $x^3 + y^4$ "E_6 singularity"

In summary, what we have found are Thom's seven elementary catastrophes

Label	$A_1^{\pm}$	A_2	$A_3^{\pm}$	A_4	$A_5^{\pm}$	$D_4^{\pm}$	D_5	(related to reflection groups)
	$\pm x^2 \pm y^2$	$x^3 \pm y^2$	$\pm x^4 \pm y^2$	$x^5 \pm y^2$	$\pm x^6 \pm y^2$	$x^2y \mp y^3$	$x^2y + y^4$	
Codim	0	1	2	3	4	3	4	

They are at the core of the so called elementary catastrophes.