Montaldi Chapter 7: Unfoldings & Catastrophes

- When a f^n w/ a degenerate critical pt is perturbed, the new f^n may have several critical pts near the older degenerate one.

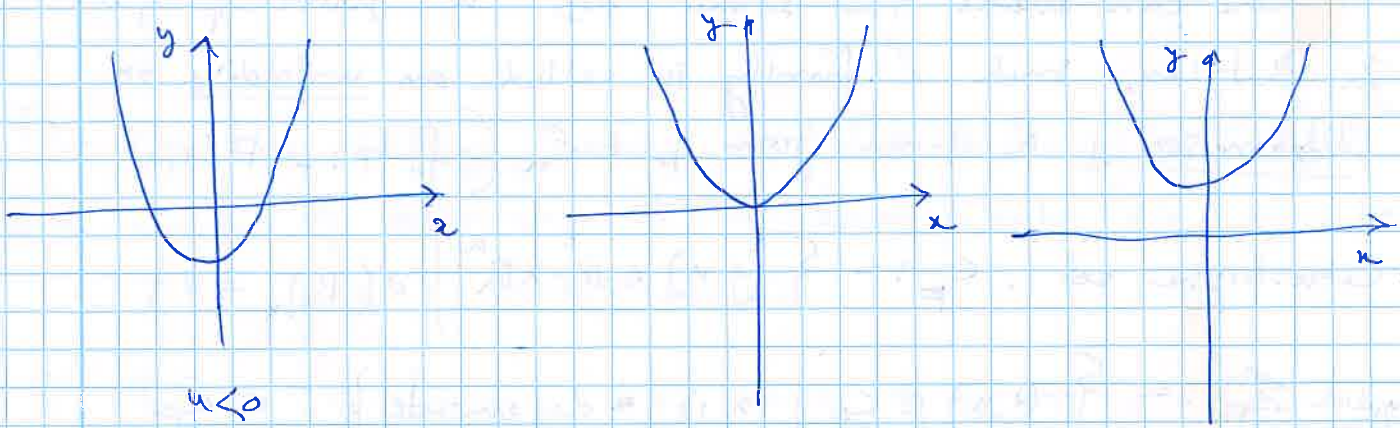
Eg: Consider the family of f^ns $f_u(x) = x^3 + ux$; "fold family"

i.e, $F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$: $F(x;u) = f_u(x) = x^3 + ux$.

Note: for $u=0$ (i.e the unperturbed case), the f^n $f_0(x) = x^3$ has a critical point of type A^2 at 0

[Recall, def^n: A critical pt p of a C^∞ f^n f of a single variable is of type A^k if $f^{(i)}(p) = 0$ $\forall i \leq k$, but $f^{(k+1)}(p) \neq 0$.]

$f_u'(x) = 3x^2 + u$. Let us plot the derivatives:

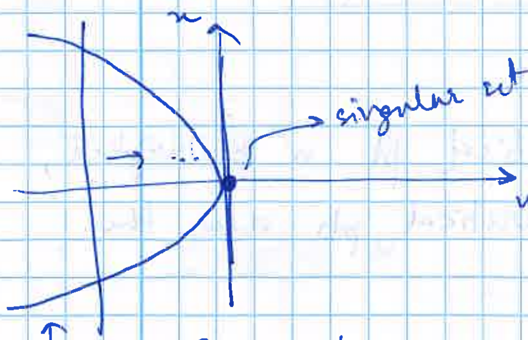


$u<0$ — 2 critical pts

$u=0$ — 1 critical pt

$u>0$ — 0 critical pt.

Behaviour of the critical pts wrt to the parameter:



: graph of $3x^2+u=0$

$C_F := \{(x,u) \mid f_u'(x) = 3x^2+u=0\}$

↳ catastrophe set.

Let us formalize & generalize this idea:

Def^n:- A C^∞ a-parameter family of f^ns on $\mathbb{R}^n$ is a C^∞ map (domain including 0) ⟶ [Note that this assumption is made by $F:\mathbb{R}^n\times\mathbb{R}^a \to \mathbb{R}$, we usually mean $F: \mathcal{U}\times V \to \mathbb{R}$, $\mathcal{U} \underset{open}{\subset} \mathbb{R}^n$, $V \underset{open}{\subset} \mathbb{R}^a$]

$$F:\mathbb{R}^n\times\mathbb{R}^a \longrightarrow \mathbb{R}$$
$$(x,u) \longmapsto f_u(x)$$

$u \to$ parameter
$x \to$ state variable.

One can make the same def^n w/ germs of f^ns.
In that case, such a family is called an unfolding or deformation of a given germ $f_0 \in \mathcal{E}_n$ ($f_0(x) = F(x,0)$).

Catastrophe set : $C_F := \{(x,u)\in\mathbb{R}^n\times\mathbb{R}^a \mid d(f_u)_x = 0\}$

Singular set $\Sigma_F := \{(x,u)\in C_F \mid x \text{ is degenerate}\} \subset C_F$

$\rightsquigarrow \pi_F : C_F \longrightarrow \mathbb{R}^a$ projection map onto parameter space
$(x,u) \longmapsto u$

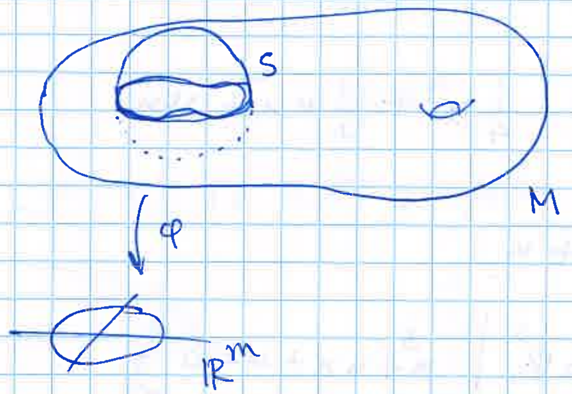
Bifurcation set $\Delta_F := \pi_F(\Sigma_F) = \{u \in \mathbb{R}^a \mid (x,u)\in\Sigma_F\}$
↳ set of parameters that induce "qualitative change" in the behaviour of the deformed functions.

Recall:

Def^n (Submanifold of $\mathbb{R}^n$) : $M \subset \mathbb{R}^n$ is a C^∞ submanifold of dim m if for any $p \in M$, $\exists$ open nbhd $S \subset \mathbb{R}^n$ of p, & an open subset $C = W \times G \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n$ & a C^∞-diffeomorphism

$$\varphi : S \to W \times G \quad \text{s.t.} \quad \varphi(M \cap S) = W \times 0_{n-m}.$$

The datum (φ, S) is called a chart.



Prop^n:- $C_F \setminus \Sigma_F$ is a C^∞ submanifold of $\mathbb{R}^n \times \mathbb{R}^a$ of dim a.

↳ Let $(x_0, u_0) \in C_F \setminus \Sigma_F$ i.e. f_{u_0} has a non-degenerate critical pt at x_0.

Then, by the splitting lemma, $\exists$ a nbhd $(x_0, u_0) \in \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^a$ & a change of coordinates of the form $(x, u) \mapsto (X(x,u), u)$ s.t

$$F(X, u) = Q(X) + h(u)$$

w/ $Q(X) = \frac{1}{2} X^T H X$, H = Hessian of f_{u_0} at x_0.

In these coordinates,

$$C_F \cap \mathcal{U} = \{ (X, u) \mid X = 0, \ (0, u) \in \mathcal{U} \}$$

$$\therefore \ \pi_F : C_F \cap \mathcal{U} \to \mathbb{R}^a, \quad (0, u) \mapsto u,$$ which is a diffeomorphism.

Note, that we have also proved.

Prop^n :- $(x,u) \in C_F \setminus \Sigma_F \implies \pi_F$ is a local diffeo around (x,u)

Q: What about the converse?

↳ Also true, we will see later!

Eg: Cusp family:

$$F(x;u,v) = f_{u,v}(x) = \frac{1}{4}x^4 + \frac{1}{2}ux^2 + vx.$$

$$\implies f'_{u,v}(x) = x^3 + ux + v$$

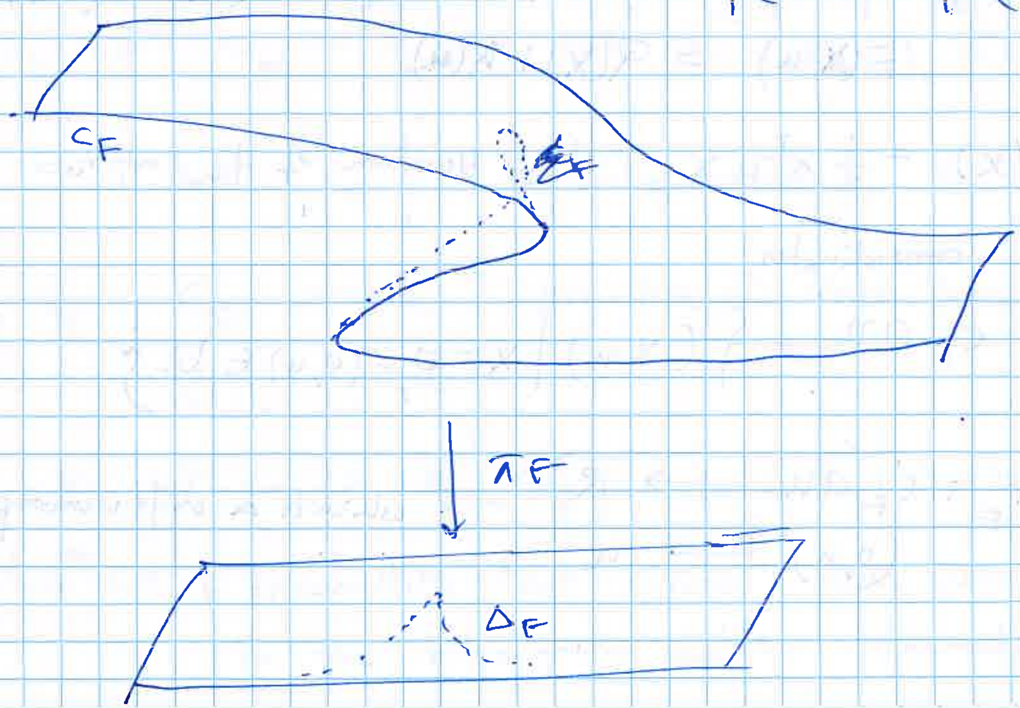
$$\therefore C_F = \{(x,u,v) \in \mathbb{R}^3 \mid x^3 + ux + v = 0\}$$

$$H_f = f''_{u,v}(x) = 3x^2 + u$$

$$\therefore \Sigma_F = \{(x,u,v) \in C_F \mid 3x^2 + u = 0\}$$

$$\therefore \Sigma_F = \{(x,u,v) \in \mathbb{R}^3 \mid u = -3x^2,\ v = 2x^3\}$$

$$\therefore \Delta_F = \{(u,v) \in \mathbb{R}^2 \mid \left(\frac{u}{3}\right)^3 + \left(\frac{v}{2}\right)^2 = 0\}$$



C_F is still a submanifold.

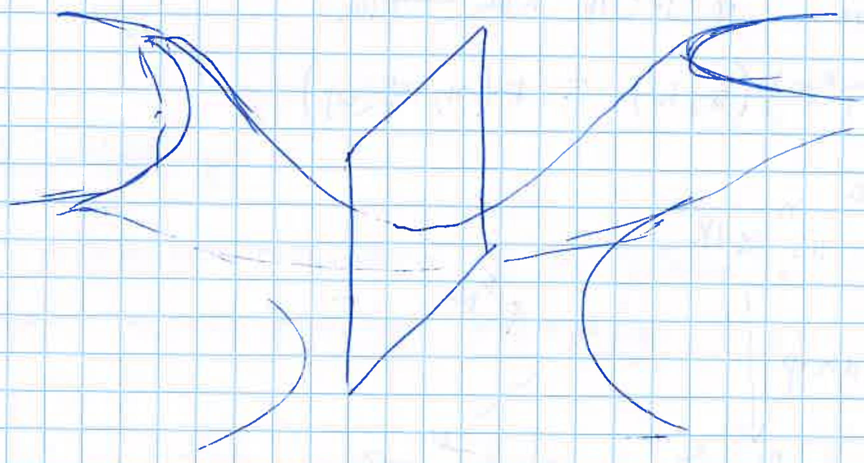
Let us see a non-eg. of that: tweak the above example a bit

$$f_{u,v}(x) = \frac{1}{4}x^4 + \frac{1}{3}ux^3 + \frac{1}{2}vx^2$$

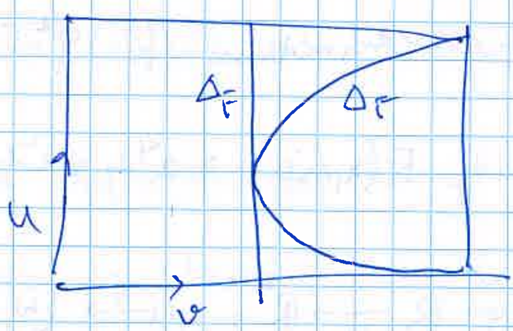
$$\Rightarrow f'_{u,v}(x) = x^3 + ux^2 + vx = x(x^2 + ux + v)$$

$$\therefore C_F = \underbrace{\{x = 0\}}_{\text{smooth}} \cup \underbrace{\{x^2 + ux + v\}}_{\text{smooth}} \subset \mathbb{R}^3$$

but C_F is not smooth.



$\downarrow$



$$\Sigma_F = \{(x,u,v) \in C_F \mid 3x^2 + 2ux + v = 0\}$$

$$= \{(0,u,0) \in \mathbb{R}^3 \mid u \in \mathbb{R}\}$$

$$\cup \{(x, -2x, x^2) \mid x \in \mathbb{R}\}$$

$$\Delta_F = \{(u,0)\} \cup \{(u,v) \mid u^2 = 4v\}$$

~~Equivalence of unfolding maps:~~

Next goal: Find out a "good" class of unfoldings for which C_F is always a submanifold. But first, some def^{n}s :-

<u>Induced unfoldings:</u>

<u>Def^n</u>:- Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a given f^n & let $F: \mathbb{R}^n \times \mathbb{R}^a \longrightarrow \mathbb{R}$ be a given unfolding of f. Now, let $\varphi: \mathbb{R}^b \longrightarrow \mathbb{R}^a$ be a given (C^∞?) map w/ $\varphi(0) = 0$. <u>The unfolding of f induced from F by φ</u> written $\varphi^* F$ is def^{n}ed to be: $\varphi^* F : \mathbb{R}^n \times \mathbb{R}^b \longrightarrow \mathbb{R}$

$$(\varphi^* F)(x; w) := F(x, \varphi(w)),$$

i.e.,

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^b \arrow[d, "id \times \varphi"'] \arrow[dr, dashed, "\varphi^* F"] & \\ \mathbb{R}^n \times \mathbb{R}^a \arrow[r, "F"'] & \mathbb{R} \end{tikzcd}$$

<u>Rmk</u>:- All this can be stated for germs at 0 as well; one needs to check the well-definedness of $\varphi^* F$ in that case.

<u>Eg</u>:- Consider the unfolding $F(x; u, v) = x^4 + ux^2 + vx$ of the map $f(x) = x^4$.
Consider the map $\varphi: \mathbb{R} \longrightarrow \mathbb{R}^2$; $w \longmapsto (w, 0)$
Then, $(\varphi^* F)(x, w) = F(x, w, 0) = x^4 + wx^2$

Q:- How are C_G, C_F & Δ_G, Δ_F etc. related?

Equivalence of unfoldings:

Def^n:- Two families $F, G : \mathbb{R}^n \times \mathbb{R}^a \longrightarrow \mathbb{R}$ are equivalent if $\exists$ a diffeomorphism $\Phi : \mathbb{R}^n \times \mathbb{R}^a \longrightarrow \mathbb{R}^n \times \mathbb{R}^a$ of the form

$$\Phi(x,u) = \big(\varphi(x,u), \psi(u)\big)$$

~~...~~ & a function germ $c : \mathbb{R}^a \longrightarrow \mathbb{R}$ s.t.

$$F(x,u) = G\big(\varphi(x,u), \psi(u)\big) + c(u).$$

This is called $\mathcal{R}^+_{un}$-equivalence

Rmk:- For each u, $G_{\psi(u)}$ is $\mathcal{R}^+$-eqv. to F_u, via the change of coordinates φ_u ; $\varphi_u(x) = \varphi(x,u)$ & the const $c(u)$.

Versal unfoldings:

idea: a versal unfolding of a germ f_0 contains all the information of all possible unfoldings of f_0.

Def^n:- Let $f_0 \in \mathcal{E}_n$ be a f^n germ, and $F : (\mathbb{R}^n \times \mathbb{R}^a, (0,0)) \to \mathbb{R}$ be an unfolding of f_0. The unfolding is versal if given any other unfolding $G : (\mathbb{R}^n \times \mathbb{R}^b, (0,0)) \longrightarrow \mathbb{R}$ of f_0, $\exists$ a map germ
$\varphi : (\mathbb{R}^b, 0) \to (\mathbb{R}^a, 0)$ s.t. G is $\mathcal{R}^+_{un}$-eqv. to $\varphi^* F$.

Q:- Nice definition, but how to actually get such an unfolding?

↳ Answered by Thom ; he gave an algebraic characterization of versal unfoldings; which leads to a way of constructing a versal unfolding of any germ of finite codimension.

algebraic characterization:

Given an unfolding ~~[illegible]~~ $F : \mathbb{R}^n \times \mathbb{R}^b \to \mathbb{R}$, the initial speeds of the unfolding are dfd to be

$$\dot{F}_j(x) := \frac{\partial F}{\partial u_j}(x,0) \quad \text{for } j = 1, \dots, b.$$

$\dot{F}_j \in \mathcal{E}_n$. Let $\dot{F} \subset \mathcal{E}_n$ be the vector subspace spanned by the initial speeds i.e.,

$$\dot{F} := \mathbb{R}\{\dot{F}_1, \dots, \dot{F}_b\}$$

Eg: if $F(x,y;u_1,u_2) = x^3 - y^2 + u_1 x - u_2^2 y - u_2 xy$, then,

$\dot{F}_1 = x$, $\dot{F}_2 = -xy$ & so $\dot{F} = \mathbb{R}\{x, xy\} \subset \mathcal{E}_2$, a 2-dim subsp.

Th^m (Versal unfoldings): Let $F(x;u)$ be an a-parameter unfolding of $f \in \mathcal{E}_n$. Then F is versal $\iff$ $\mathcal{E}_n = Jf + \mathbb{R} + \dot{F}$

↳ Proof involves Malgrange preparation th^m; will be discussed by Jinkun next week.

- we can take the initial speeds s.t all $\dot{F}_j \in \mathfrak{m}_n$; in that case the cond^n for versality can be replaced by

$$\mathfrak{m}_n = Jf + \dot{F}.$$

i.e for F to be versal, some of the initial speeds form a cobasis for Jf in $\mathfrak{m}_n$.

Cor: If $F : \mathbb{R}^n \times \mathbb{R}^a \to \mathbb{R}$ is a versal unfolding of f, then, $a \geq \operatorname{codim}(f)$.

Cor:- Let $f \in \mathcal{E}_n$ be a germ of finite codimension r, & let $h_1, \ldots, h_r$ be a cobasis for Jf in m_n. Then,

$$F(x;u) = f(x) + u_1 h_1(x) + \ldots + u_r h_r(x)$$

is a versal unfolding of f.

$(a > n\ ??)$

Prop^n:- Let $F : (\mathbb{R}^n \times \mathbb{R}^a, (0,0)) \to \mathbb{R}$ be a versal unfolding of a germ $f \in \mathcal{E}_n$. Then:

(i) C_F is a submanifold of $\mathbb{R}^n \times \mathbb{R}^a$ of dim a.

(ii) $(x,u) \in \Sigma_F \iff \chi_u$ is <u>NOT</u> a ~~diffeo~~ local diffeo at (x,u).

↳ (i) Suppose $f \in m_n^3$ (otherwise use splitting lemma).

Then the monomials $x_1, \ldots, x_n$ are part of the cobasis of Jf.

So, one versal unfolding is of the form:-

$$G(x;u,v) = G_1(x;u) + \sum_{j=1}^{n} v_j x_j$$

where $v \in \mathbb{R}^n$, $u \in \mathbb{R}^{a-n}$, G_1 involves the remaining unfolding terms.

Then,

$$C_G = \left\{ (x,u,v) \in \mathbb{R}^n \times \mathbb{R}^{a-n} \times \mathbb{R}^n \;\middle|\; v_j = -\frac{\partial G_1}{\partial x_j}(x,u),\ j = 1, \ldots, n \right\}.$$

This a-dim submanifold, ∵ its a graph of the C^∞ f^n

$$\mathbb{R}^n \times \mathbb{R}^{a-n} \to \mathbb{R}^n : (x,u) \mapsto \left(-\frac{\partial G_1}{\partial x_1}(x,u), \ldots, -\frac{\partial G_1}{\partial x_n}(x,u) \right)$$

For the given unfolding F, ∃ a diffeo $\mathbb{R}^n \times \mathbb{R}^a$ ⟲ which maps C_G to C_F,

∴ C_F is also a submfld

~~Eg: Let $f(x,y) = x^2 + xy^3$, $Jf = \langle 2x+y^3, 3xy^2\rangle$~~

~~Eg: (Cusp family):~~

~~$f(x) = x^4$, $\frac{\partial f}{\partial x} = 4x^3$~~

Eg: (Cusp family):

$$f(x) = x^4, \quad \frac{\partial f}{\partial x} = \langle x^3 \rangle$$

$$\therefore \text{cobasis}(f) = \{x, x^2\}$$

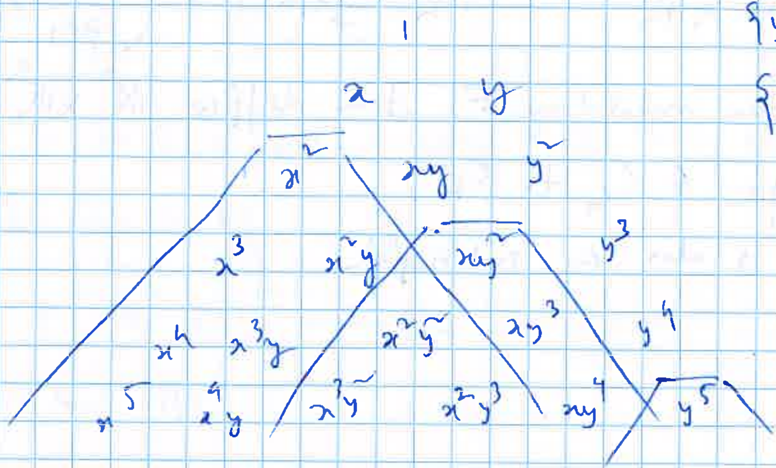
$\therefore$ A miniversal unfolding is the cusp family we talked about earlier:

$F(x, u, v)$ $f_{u,v}(x) = x^4 + ux^2 + vx$

Recall C_F was smooth!

Eg: Let $f(x,y) = x^2 + xy^3$, $Jf = \langle 2x+y^3, xy^2\rangle$

Newton's diagram for f:



Possible cobasis:

$\{y, y^2, y^3, y^4\}$

$\{x, y, xy, y^2\}$