Catastrophies with symmetries

Annika Thiele

Certain problems, to which we can apply Catastrophe Theory, give rise to symmetries naturally. This motivates modifying the theory we have learned so far in this seminar to functions obeying symmetry conditions. In this talk, we will look at functions that are invariant under some group action. In particular, we will concentrate on even functions, invariant under the non-trivial $\mathbb{Z}/2\mathbb{Z}$ action.

1 Invariant Theory

The following discussion follows closely chapters 9, 16.2 and 16.4 of [Mon21] and some of chapters 12.4 and 12.6 of [GSS88].

Let us recall what the representation of a group on a vector space is.

Definition 1.1 (Representations and Actions)**.** A group G acts (linearly) on a vector space V if there exists a mapping

$$G \times V \to V, \qquad (g, v) \mapsto g \cdot v,$$

such that the following conditions hold:

1. The map $\rho_g : V \to V, v \mapsto g \cdot v$ is linear. We call the map $\rho : G \to \hom(V, V), g \mapsto \rho_g$ a Representation of G on V.
2. For any two group elements $g_1, g_2 \in G$ and any $v \in V$,

$$g_1(g_2 \cdot v) = (g_1 g_2) \cdot v.$$

Morally, an action tells us how a group element transforms a vector and a representation tells us how a group element transforms the entire space.

Example 1.2.

1. The group $G = \mathbb{Z}/2\mathbb{Z}$ acts non-trivially on $V = \mathbb{R}^n$ by $-1 \cdot x = -x$ for all $x \in \mathbb{R}^n$.
2. The group $G = S^1$ acts on $\mathbb{R}^2 \cong \mathbb{C}$ by $\theta \cdot z = e^{i\theta} z$.

Definition 1.3 (Invariant functions)**.** Let G be a compact Lie group, e.g. a finite group, acting on an $\mathbb{R}$-vector space V, a function $f : V \rightarrowtail \mathbb{R}$ is invariant under G if

$$f(g \cdot x) = f(x) \quad \forall g \in G, \;\; x \in \mathrm{dom}(f)$$

An invariant polynomial is an invariant function that is polynomial.

Example 1.4.

1. The first example of invariant functions are the *even functions*. Consider the non-trivial action of $G = \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}^n$. A function is called even if it is $\mathbb{Z}/2\mathbb{Z}$ invariant, i.e for all $x \in \mathbb{R}^n$, $f(-x) = f(x)$. We will later modify the theory to this class of functions. The term *even* makes sense when looking at polynomials in one variable, then a polynomial is even if and only if it has only even exponents.

2. Consider the action of S^1 on $\mathbb{R}^2 \cong \mathbb{C}$ described above. Then $f : \mathbb{C} \to \mathbb{R}$ is S^1-invariant if and only if $f(\theta \cdot z) = f(z)$ holds for all $\theta \in S^1$ and complex numbers z. That means, f is constant on circles centered at the origin.

Instead of functions, we can also look at the germs of G-invariant functions. As in the general case these admit a ring structure.

Definition 1.5. We denote the ring of smooth G-invariant germs $f : (V, 0) \to \mathbb{R}$ as $\mathcal{E}(G)$. If $V = \mathbb{R}^n$, we write $\mathcal{E}_n(G)$ and if additionally, $G = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{E}_n^+$ labels the ring of smooth even germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}$. Finally, $\mathcal{P}(G)$ is the ring of invariant polynomials.

Similar to the general case, $\mathcal{E}_n^+$ is a local ring with the unique maximal ideal $\mathfrak{m}_n^+$ consisting of germs fixing the origin. It has the same generators as $\mathfrak{m}_n^2$, namely the monomials of degree two:

$$\mathfrak{m}_n^+ = \langle x_1^2, x_1x_2, \ldots, x_n^2 \rangle,$$

but $\mathfrak{m}_n^+$ and $\mathfrak{m}_n^2$ are ideals over different rings and hence not the same. For example, $x_1^3 \in \mathfrak{m}_n^2 \setminus \mathfrak{m}_n^+$. But we can say $\mathfrak{m}_n^+ = \mathfrak{m}_n \cap \mathcal{E}_n^+ = \mathfrak{m}_n^2 \cap \mathcal{E}_n^+$.
Even polynomials have only even exponents, but can we generalize odd polynomials (with only odd exponents) to a notion on smooth functions? Realize that a polynomial is odd if and only if $f(-x) = -f(x)$, a condition that is very similar to the even case.

Definition 1.6. A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is odd if it satisfies $f(-x) = -f(x)$ for all $x \in \mathbb{R}^n$. Then write $\mathcal{E}_n^-$ for the set of odd smooth function germs.

Achtung: $\mathcal{E}_n^-$ is NOT a ring, but it is an $\mathcal{E}_n^+$-module.

Lemma 1.7. *The ring of smooth germs admits a direct sum splitting* $\mathcal{E}_n = \mathcal{E}_n^+ \oplus \mathcal{E}_n^-$.

Proof. Consider a smooth germ $f \in \mathcal{E}_n$. We can write $f = f^+ + f^-$ where $f^+ := \frac{1}{2}(f(x) + f(-x))$ and $f^- := \frac{1}{2}(f(x) - f(-x))$. Since f^+ is even and f^- is odd, $\mathcal{E}_n = \mathcal{E}_n^+ + \mathcal{E}_n^-$. The sum is direct, since $\mathcal{E}_n^+ \cap \mathcal{E}_n^- = \{0\}$. □

In the following we will use two results that we will not prove, but proofs can be found in [GSS88] and [NS10].

Result 1.8 (Hilbert-Weyl Theorem)**.** *Let G be a compact Lie group acting on V. Then there exists a finite Hilbert basis of $\mathcal{P}(G)$, i.e. there are finitely many G-invariant polynomials $p_1, \ldots, p_k \in \mathcal{P}(G)$ such that every $f \in \mathcal{P}(G)$ can be written as a polynomial of $p_1, \ldots, p_k$.*

Example 1.9. $\{x^2\}$ is a Hilbert basis for the ring of even polynomials in one variable. Indeed, given any even polynomial $f(x) = a_0 + a_1 x^2 + \cdots + a_n x^{2n}$, we can write $f(x) = g(x^2)$ for $g(x) = a_0 + a_1 x + \cdots + a_n x^n$.

We are working towards proving Schwartz's Theorem, which generalizes this result to the ring of G-invariant smooth germs. The proof we give will require another result stating that the ideal generated by the elements of the Hilbert basis is of finite codimension in the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$. This fails if G is not finite.

Result 1.10 (Emmy Noether). *Let G be finite. $p_1, \ldots, p_k$ be a Hilbert basis of $\mathcal{P}(G)$, define the function $\phi : V \to \mathbb{R}^k, x \mapsto (p_1(x), \ldots, p_k(x))$. Then the ideal $I_\phi := \langle p_1, \ldots, p_k \rangle$ is of finite codimension in the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$, where $n = \dim(V)$.*

We can now prove the generalization of the Hilbert-Weyl Theorem to all G-invariant smooth germs.

Theorem 1.11 (Schwartz). *Let G be a compact Lie group acting on $\mathbb{R}^n$. Moreover, let $f \in \mathcal{E}_n(G)$ and $p_1, \ldots, p_k$ be a Hilbert basis of $\mathcal{P}(G)$. Then there exists a smooth germ $h \in \mathcal{E}_k$ such that*

$$f(x) = h(p_1(x), \ldots, p_k(x)).$$

Remark. This recovers the Fundamental Theorem of Whitney, since $\{x^2\}$ is a Hilbert basis of $P(\mathbb{Z}/2\mathbb{Z})$ in one variable:

Theorem 1.12 (Fundamental Theorem of Whitney). *A smooth function $f : \mathbb{R} \rightarrowtail \mathbb{R}$ is even if and only if there exists a function $g : \mathbb{R} \rightarrowtail \mathbb{R}$ such that $f(x) = g(x^2)$ for all $x \in dom(f)$.*

Although we have stated Schwartz's theorem in the general case for any compact Lie group, we will only prove it for finite groups G. A proof of the general theorem can be found in [GSS88].

Proof. The proof will use the Malgrange-Mather preparation theorem, of which we have seen the proof in a previous talk, so let us recall the statement.

Theorem 1.13 (Magrange-Mather preparation theorem). *Let $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be the germ of a smooth map and A a finitely generated $\mathcal{E}_n$-module for which $A/I_\phi A$ is finite dimensional. Then A is also finitely generated as an $\mathcal{E}_k$-module. More precisely, if $\{u_0, \ldots, u_r\} \subset A$ is a cobasis for $I_\phi A$, then it is also a generating set for A as an $\mathcal{E}_p$-module. This means, that each $a \in A$ can be written as $a = (h_0 \circ \phi)u_0 + \cdots + (h_r \circ \phi)u_r$ for some $h_0, \ldots, h_r \in \mathcal{E}_k$.*

By Result 1.10, the ideal I_ϕ is of finite codimension in the ring of polynomials $\mathbb{R}[x_1, \ldots, x_n]$.

Lemma 1.14. *We can choose a cobasis $\{u_0, \ldots, u_r\}$ of I_ϕ such that $u_0 = 1$ and for all $j > 0$, u_j has average zero, i.e.*

$$\frac{1}{|G|} \sum_{g \in G} u_j(g \cdot x) = 0,$$

for all $x \in \mathbb{R}^n$.

We will postpone the proof of the lemma to the end of this proof. So choose $\{u_0, \dots, u_r\}$ to be such a cobasis of I_ϕ. Now let us define the ideal $J_\phi := \langle p_1, \dots, p_k \rangle \lhd \mathcal{E}_n$ with the same generators as I_ϕ, but over the ring of smooth germs. Then $\{u_0, \dots, u_r\}$ is also a cobasis of J_ϕ in $\mathcal{E}_n$ ([Mon21] [Remark 3.14]). In particular, J_ϕ is of finite codimension in $\mathcal{E}_n$. Applying Theorem 1.13 to $A = \mathcal{E}_n$ and ϕ shows that any $f \in \mathcal{E}_n$ can be written as

$$f = \sum_{j=0}^{r} u_j (h_j \circ \phi),$$

for some $h_0, \dots, h_r \in \mathcal{E}_k$. If f is G-invariant,

$$f(x) = f(g \cdot x) = \sum_{j=0}^{r} u_j(g \cdot x) \cdot h_j(\phi(x)),$$

since ϕ is G-invariant by definition. Summing over $g \in G$ yields

$$\begin{aligned} |G| f(x) &= \sum_{j=0}^{r} \left(\sum_{g \in G} u_j(g \cdot x) \right) h_j(\phi(x)) \\ &= h_0(\phi(x)). \end{aligned}$$

The second equality holds by choice of the u_j. This proves the statement. □

It remains to prove the lemma.

Proof of Lemma 1.14. Start with any cobasis $\{v_0, \dots, v_r\}$ such that $v_0 = 1$ and $v_j \in \mathfrak{m}_n$ for all $j > 0$. The function

$$\rho_j(x) := \frac{1}{|G|} \sum_{g \in G} v_j(g \cdot x)$$

is G-invariant. Indeed, for any element $h \in G$

$$\begin{aligned} \rho_j(h \cdot x) &= \frac{1}{|G|} \sum_{g \in G} v_j(gh \cdot x) \\ &= \frac{1}{|G|} \sum_{k \in G} v_j(k \cdot x) \\ &= \rho_j(x), \end{aligned}$$

using $G = \{gh | g \in G\}$. Result 1.8 implies that $p_j \in I_\phi$. Set $u_0 = v_0 = 1$, $u_j = v_j - \rho_j$, then $\{u_0, \dots, u_r\}$ is a cobasis of I_ϕ and satisfies the wanted conditions. □

Remark. If G is not finite, I_ϕ does not have to be of finite codimension. Thus, this argument fails.

2 Modify Theory to Even Functions

Let's recap what we have introduced so far. $\mathcal{E}_n^+$ is the ring of even smooth germs with unique maximal ideal $\mathfrak{m}_n^+ = \langle x_1^2, x_1x_2, \ldots, x_n^2 \rangle$. The set of odd smooth germs is denoted by $\mathcal{E}_n^-$.

2.1 Right-equivalence

Let us adapt the notion of right-equivalence to even functions, by requiring the change of coordinates to be odd. Precisely,

Definition 2.1. We say two even function germs $f, g \in \mathcal{E}_n^+$ are $\mathcal{R}^{ev}$-equivalent if there is a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ satisfying $\phi(-x) = -\phi(x)$, such that $f = g \circ \phi$.

Remark. Choosing ϕ to be odd really does perserve evenness:

$$(f \circ \phi)(-x) = f(\phi(-x)) = f(-\phi(x)) = f(\phi(x)) = (f \circ \phi)(x).$$

Lemma 2.2. *If θ_n^- is the set of odd vector fields, $\theta_n^- \subseteq \mathfrak{m}_n\theta_n$.*

Proof. Let $v \in \theta_n^-$, then v(-x)=-v(x). Let $x = 0$, then $v(0) = -v(0)$, so $v(0) = 0$. □

Definition 2.3. The even Jacobian ideal of an even function germ $f \in \mathcal{E}_n^+$ is defined as

$$J^+f := tf(\theta_n^-) := \left\{ \sum_j \frac{\partial f}{\partial x_j} v_j \mid v_j \in \mathcal{E}_n^- \right\}.$$

Lemma 2.4. $J^+f = Jf \cap \mathcal{E}_n^+$.

Proof. For an even function germ f, the partial derivatives $\frac{\partial f}{\partial x_i}$ are odd by the chain rule. Define $g(x) = -x$, then by evenness of f, $f \circ g = f$:

$$\frac{\partial f}{\partial x_i}(-x) = -\frac{\partial f}{\partial x_i}(-x)\frac{\partial g}{\partial x_i}(x) = -\frac{\partial (f \circ g)}{\partial x_i}(x) = -\frac{\partial f}{\partial x_i}(x).$$

Hence, each element $\sum_j \frac{\partial f}{\partial x_j} v_j \in J^+f$ is even and the inclusion $J^+f \subseteq Jf \cap \mathcal{E}_n^+$ holds. For the other inclusion, notice if $\sum_j \frac{\partial f}{\partial x_j} v_j \in Jf$ is even, $v_j \in \mathcal{E}_n^-$, since the partial derivatives of f are odd. □

Remark. Since the monomials of order one generate $\mathcal{E}_n^-$ as a $\mathcal{E}_n^+$-module, $J^+f = \langle x_i \frac{\partial f}{\partial x_j} \rangle_{i,j}$.

In the general theory, the finite determinacy theorem gave us a condition, when a function f is finitely determined, i.e for some k, the equality of the k-jets of f and some other germ g implies the right-equivalence of the two. Similarly, one can ask oneself, when an even function germ is finitely determined with respect to our new notion of right equivalence for even germs. The answer is given by the following theorem.

Theorem 2.5 (Finite determinacy theorem for $\mathcal{R}^{ev}$-equivalence). *A function germ $f \in \mathcal{E}_n^+$ is $2k$-determined with respect to $\mathcal{R}^{ev}$-equivalence, if $(\mathfrak{m}^+)^{k+1} \subseteq \mathfrak{m}_n^+ J^+ f$.*

Proof (Sketch). Notice that $(\mathfrak{m}_n^+)^{k+1} = \mathfrak{m}_n^{2k+1} \cap \mathcal{E}_n^+$ and $\mathfrak{m}_n^+ J^+ f = \mathfrak{m}_n^2 Jf \cap \mathcal{E}_n^+$. Now assume $\mathfrak{m}_n^{2k+1} \cap \mathcal{E}_n^+ \subseteq \mathfrak{m}_n^2 Jf \cap \mathcal{E}_n^+$. The proof will follow along the lines of the proof of the Finite Determinacy Theorem, with some minor alterations. Said proof used the homotopy method, so similarly define the homotopy $f_s = f + sh$ but now for $h \in \mathfrak{m}_n^{2k+1} \cap \mathcal{E}_n^+$. We want to construct a family of odd diffeomorphisms ϕ_s satisfying $f_s \circ \phi_s = f$. We find this by solving the infinitesimal homotopy equation,

$$d(f_s)_y(v_s(y) = -h(y),$$

(with $y = \phi_s(x)$) for all s. We can solve the infinitesimal homotopy equation as in the general case. Let v_s be a solution. Similar to Lemma 1.7, $\theta_n = \theta_n^+ \oplus \theta_n^-$ and we can write $v_s = v_s^+ + v_s^-$. One checks that if v_s solves the infinitesimal homotopy equation, so does v_s^-. Now take the flow of this vector field and show that it is odd (since v_s^- is odd). In this manner, we get the family ϕ_s of odd diffeomorphisms we are looking for. □

2.2 Unfolding and Versality

Definition 2.6. We define the even codimension of $f \in \mathfrak{m}_n^+$ to be

$$\operatorname{codim}^+(f) := \dim(\mathfrak{m}_\mathfrak{n}^+ / J^+ f)$$

Example 2.7.

1. Consider $f(x) = \sum_i x_i^2$. Then

   $$J^+ f = \langle x_i x_j \rangle_{i,j} = \mathfrak{m}_n^+,$$

   so $\operatorname{codim}^+(f) = 0$.

2. For $f(x, y) = x^4 + y^4$, we can compute the even Jacobian ideal,

   $$J^+ f = \langle x^4, x^3 y, xy^3, y^4 \rangle.$$

   A cobasis of $J^+ f$ in $\mathfrak{m}_2^+$ is given by $\{x^2, xy, y^2, x^2y^2\}$ (see Figure 1), so the even codimension is $\operatorname{codim}^+(f) = 4$.

When looking at unfoldings of an even function, we can require that the smooth family $F : \mathbb{R}^n \times \mathbb{R}^a \to \mathbb{R}$ is a family of even functions, i.e. f_u is even for all $u \in \mathbb{R}^a$. Then we consider versality among these families. The next theorem states a condition for this versality.

Theorem 2.8. *A smooth family $F : \mathbb{R}^n \times \mathbb{R}^a \to \mathbb{R}$ of even functions is versal among the class of smooth families of even functions if and only if*

$$J^+ f + \mathbb{R} + \dot{F} = \mathcal{E}_n^+,$$

here $\dot{F} := \langle \dot{F}_i \rangle$ is the ideal generated by the inition speeds of the unfolding.

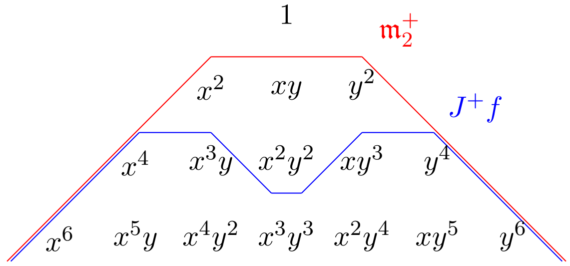

Figure 1: Newton-diagram for Example 2.7

Example 2.9.

1. Let $f(x, y) = x^6 + y^4 \in \mathcal{E}_2^+$. Then the even Jacobian ideal is given by

$$J^+f = \langle x^6, x^5y, xy^3, y^4 \rangle$$

Notice that

$$(\mathfrak{m}_2^+)^4 = \langle x^8, x^7y, x^6y^2, x^5y^3, x^4y^4, x^3y^5, x^2y^6, xy^7, y^8 \rangle \subset J^+f,$$

so Theorem 2.5 implies that f is 8-determined with respect to $\mathcal{R}^{ev}$-equivalence. Moreover, $\{x^2, xy, y^2, x^4, x^3y, x^2y^2, x^4, y^4\}$ is a cobasis of J^+f. So $\operatorname{codim}^+(f) = 7$ and by Theorem 2.8,

$$F(x, y, u_1, \ldots, u_7) = x^6 + y^4 + u_1x^2 + u_2xy + u_3y^2 + u_4x^4 + u_5x^3y + u_6x^2y^2 + u_7x^4y^2$$

is a versal unfolding of f among the class of smooth familes of even functions.

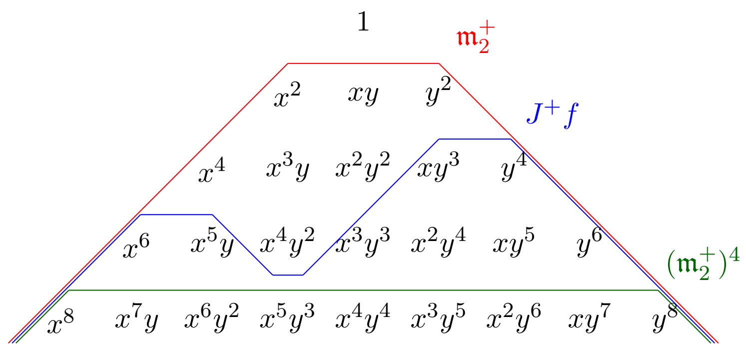

Figure 2: Newton-diagram for Example 2.9

2. Similar in the case of three variables, consider $f(x, y, z) = x^4 + y^4 + z^2$. The even Jacobian ideal ist

$$J^+f = \langle x^4, x^3y, x^3z, xy^3, y^4, y^3z, xz, yz, z^2 \rangle.$$

A cobasis of $J^+ f$ is $\{x^2, xy, y^2, x^2y^2\}$, so $\text{codim}^+(f) = 4$ and

$$F(x, y, z, r, s, t, u) = x^4 + y^4 + z^2 + rx^2 + sxy + ty^2 + ux^2y^2$$

is versal.

References

[GSS88] Martin Golubitsky, Ian Stewart, and David G Schaeffer. *Singularities and Groups in Bifurcation Theory, Vol. II.* Applied Mathematical Sciences, vol. 69. New York, NY: Springer New York, 1988.

[Mon21] James Montaldi. *Singularities, Bifurcations and Catastrophes.* Cambridge University Press, 2021.

[NS10] Mara Dicle Neusel and Larry Smith. *Invariant theory of finite groups.* Mathematical surveys and monographs, vol. 94. Providence, Rhode Island, 2010.