Bifurcation problems and contact equivalence

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This talk lays the groundwork for singularity theory of nonlinear equations. It introduces $\mathcal{K}$- and $\mathcal{C}$-equivalence. The notes present chapter 10 and 11 of James Montaldis book "Singularities Bifurcations and Catastrophes" [3] with an additional insight to $\mathcal{K}$-equivalence in respect to vector and fiber bundles on manifolds.

Bifurcation

Definition 1 (Bifurcation Problem [3, Def. 10.1])
A bifurcation problem is a smooth map $G : \mathbb{R}^n \times \mathbb{R}^k \rightarrowtail \mathbb{R}^p$ (defined on an open subset of $\mathbb{R}^n \times \mathbb{R}^k$), or it is the germ at a point of such a map. For each $u \in \mathbb{R}^k$ (the base space, or parameter space) we denote by $g_u : \mathbb{R}^n \rightarrowtail \mathbb{R}^p$ the map

$$g_u(x) := G(x; u).$$

As usual we refer to the g_u as a smooth family of maps with parameter u; the variables denoted by x are called state variables. Given such a bifurcation problem, we are interested in the solutions of the corresponding equations, so we denote the set of zeros of G by

$$Z_G := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^k | \ G(x, u) = 0\}.$$

This is called the zero-set of G. The bifurcation problem (or family of maps) G is said to be regular if as a map (germ), G is a submersion.

If the bifurcation problem is regular the zero-set Z_G is the preimage of a submersion, thus preimage of a map with surjective differential, which gives us that the preimage is a (sub-)manifold by the regular value theorem.
In most cases the dimension of the parameter space (also called state variables) p is equal to the number of equations.
Suppose we have a family of functions $F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ one may define a family of functions via the gradient

$$G := \nabla_x F.$$

In that case the zero-set Z_G equals the catastrophe set C_F of F.

Definition 2 (Singular Set of a Bifurcation Problem [3, Def. 10.2])
The singular set of a bifurcation problem Gv is

$$\Sigma_G = \{(x,u) \in Z_G |\ rank(d(g_U)(x)) < p\}.$$

In most applications $p = n$ and in that case,

$$\Sigma_G = \{(x,u) \in Z_G |\ \det(dg_U(x)) = 0\}.$$

The discriminant of bifurcation set is the subset of the base space,

$$\Delta_G = \pi_G(\Sigma_G)$$

where $\pi_G : Z_G \to \mathbb{R}^k$ is the projection $\pi_G(x,u) = u$. That is,

$$\Delta_G = \{u \in \mathbb{R}^k |\ \exists x, (x,u) \in \Sigma_G\}.$$

Example ([3, Def. 10.3.])**:**
Let $G(x,y;u) = (x^2 + y^2 - u, xy)$. One obtains

$$Z_G = \{(x,y,u) \in \mathbb{R}^3 |\ x,y = 0, u = x^2 + y^2\} \quad = \{(x,0,x^2)|\ x \in \mathbb{R}\} \cup \{(0,y,y^2)|\ y \in \mathbb{R}\}$$

which is the union of two parabolas in $\mathbb{R}^3$ and is singular at the origin. Further more $dg_u = \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix}$, such that

$$\Sigma_G = \{(x,y,u) \in Z_G |\ 2x^2 - 2y^2 = 0\} = \{(0,0,0)\}$$

and thus

$$\Delta_G = \{u = 0\}.$$

A typical source for this problem are equilibrium points of real dynamical systems.
To broaden our portfolio of tools we will establish contact equivalence to study zero-sets of systems of equations. One point of focus in our study will be versal unfolding of systems of equations and maybe in the last talk our theme will get an outlook on bifurcation problems from the perspective of contact equivalence.

Definition 3 (Singular Set of a Map [3, Def. 10.4.])
Given a map $f : \mathbb{R}^n \rightarrowtail \mathbb{R}^p$, one defines the singular set (or critical set) of f to be

$$\Sigma_f := \{x \in \mathbb{R}^n |\ rk(df_x) < p\},$$

and its discriminant Δ_f is the image $f(\Sigma_f) \subset \mathbb{R}^p$.

The definition of the singular set defined for a bifurcation problem does not equal the singular set of the underlying bifurcation map. Σ'_G the singular set of G lies in the singular set of the bifurcation problem $\Sigma'_G \subset \Sigma_G$

Contact equivalence

Remark (Notation)**:**
A matrix M is said to be in $Gl_n(\mathcal{E}_n)$ if M is a $n \times n$ matrix whose entries are in $\mathcal{E}_n$ and moreover the matrix is invertible with the entries of $M^{-1}(x)$ also belonging to $\mathcal{E}_n$.

Definition 4 (Contact Equivalence [3, Def. 11.1])
Two map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are contact equivalent or $\mathcal{K}$-equivalent, if there exists,

1. a diffeomorphism ϕ of the source $(\mathbb{R}^n, 0)$, and
2. a matrix $M \in Gl_p(\mathcal{E}_n)$ such that

$$f \circ \phi(x) = M(x)g(x),$$

where $f(x)$ and $g(x)$ are written as column vectors, and $M(x)g(x)$ is the usual product of matrix times vector.
If ϕ is the identity, one says f and g are $\mathcal{C}$-equivalent. In this case $f(x) = M(x)g(x)$.

Example:
Let $g(x, y) = (x^2, y^2)$ and $f(x, y) = (x^2 + y^2, x^2 - y^2 + y^3)$ be considered as germs at the origin. Then f and g are C-equivalent, with $M(x, y) = \begin{pmatrix} 1 & 1 \\ 1 & y-1 \end{pmatrix}$. Note that $M(x, y)$ is invertible for (x, y) in a neighborhood of $(0, 0)$

Proposition 1 (Zero-sets and Contact Equivalence[3, Prop. 11.3])
If f and g are $\mathcal{C}$-equivalent then their zero-sets are the same.
Equivalently if f and g are $\mathcal{K}$-equivalent then their zero-sets are diffeomorphic subsets of $\mathbb{R}^n$.

Remark (Converse Direction)**:**
The converse of Proposition 1, a diffeomorphic zero set does not imply contact equivalence as as can be seen in the above example with x^2 and x.

Proof. Let f and g be $\mathcal{C}$-equivalent. Since $M(x)$ is invertible,

$$f(x) = 0 \Leftrightarrow g(x) = 0$$

and if f and g are $\mathcal{K}$-equivalent one obtains that

$$g(x) = 0 \Leftrightarrow f(\phi(x)) = 0$$

□

Remark (Images of Contact Equivalent functions)**:**
Images of contact equivalent map germs also do not need to be diffeomorphic. A counterexample here are the maps $\mathbb{R} \to \mathbb{R}^2$ given by $f(t) = (t^2, 0)$ and $g(t) = (t^2, t^3)$. These are $\mathcal{K}$-equivalent but their images are not diffeomorphic.

Some Authors also call this type of equivalence the $V-$equivalence for V as in Variety, as in the book of V. I. Arnold [1] to distinguish the name from other concepts, e.g. the contact group.

Definition 5 (Local Algebra)
Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth map germ and denote by I_f the ideal in $\mathcal{E}_n$ generated by the components of f. Define the local algebra of f to be

$$Q(f) := \frac{\mathcal{E}_n}{I_f}.$$

For example, if $f(x, y) = (x^2, y^2)$ then $I_f = \langle x^2, y^2 \rangle$ and

$$Q(f) = \frac{\mathcal{E}_2}{\langle x^2, y^2 \rangle} \simeq \mathbb{R}\{1, x, y, xy\}.$$

Even if it only seems like a vector space isomorphism we have more structure since the local algebra $Q(f)$ has the ring structure inherited from $\mathcal{E}_n$.

Example ([3, Expl. 11.4.])**:**
If $h \in \mathcal{E}_n$ is the germ of a smooth function then with $f = \nabla h$ we have

$$Q(f) = \frac{\mathcal{E}_n}{J_h}$$

which is a familiar object from the Talk about $\mathcal{R}$-equivalence or [3, Chap. 4].

The importance of the local algebra I_f for $\mathcal{C}$-equivalence is highlighted by the following theorem by Mather.

Theorem 2 ([3, Thm. 11.5.])
Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two map germs. The following statements are equivalent:

1. *f and g are $\mathcal{C}$-equivalent.*
2. *Their ideals I_f and I_g are equal.*
3. *Their local algebras are equal.*

Example ([3, Expl. 11.6.])**:**
Consider $f(x, y) = (x^2, y^2)$, and $g(x, y) = (x^2 + y^3, y^2 + x^3)$. By Nakayama's lemma we may calculate $\langle x^2, y^2 \rangle = \langle x^2 + y^3, y^2 + x^3 \rangle$. Consequently $I_f = I_g$ and thus the maps are $\mathcal{C}$-equivalent. One may also construct directly an appropriate matrix M.

Lemma 3 ([3, Lem. 11.8.])
Given any pair of $p \times p$ matrices A, B with real entries, there is a matrix C (depending only on B) such that $M := C(Id_p - AB) + B$ is invertible.

Corollary 4 ([3, Thm. 11.7.])
Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two map germs. The following statements are equivalent:

1. *f and g are $\mathcal{K}$-equivalent.*

2. *Their ideals I_f and I_g are diffeomorphic in $\mathcal{E}_n$.*

3. *Their local algebras are diffeomorphic.*

Proof of Theorem 2. $(ii) \Leftrightarrow (iii)$:
$\frac{R}{I}$ is the set of equivalence classes in the ring R modulo the ideal I, and one of these classes (the zero of multiplication in Q_I) is just I. So if $\frac{R}{I} = \frac{R}{J}$ then $I = J$. And conversely.
$(i) \Leftrightarrow (ii)$:
Let f and g be $\mathcal{K}$-equivalent, then there exists a Matrix $M \in Gl(\mathcal{E}_n)$ such that

$$f(x) = M(x)g(x).$$

By matrix multiplication we can represent the i-th component of f by

$$f_i(x) = \sum_{j=1}^{n} m_{i,j} g_j$$

So each component f_i has to be in the Ideal of I_g. One may repeat the calculation with M^{-1} and gets that $g_j \in I_f$.
$(ii) \Leftrightarrow (i)$:
Let I_f and I_g be diffeomorphic. Since $f_i \in I_g$ there exist n coefficients $a_{i,j} \in I_g$ such that

$$f_i(x) = \sum_{j=1}^{n} a_{i,j} g_j,$$

and similarly there exist coefficients $b_{i,j}$ for the components of g

$$g_i(x) = \sum_{j=1}^{n} b_{i,j} f_j.$$

Combining the two identities we obtain

$$f_i = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{i,j} b_{j,k} f_k.$$

Defining the $p \times p$ matrices $A := (a_{i,j})$ and $B := (b_{i,j})$ one may use lemma 3 with $A(0)$ and $B(0)$, then the lemma tells us that there exists a matrix C such that

$$M(x) = C(Id_p - A(x)B(x)) + B(X)$$

is invertible at 0. Since M is continuous it must be invertible in a neighbourhood of 0. And it holds that

$$\begin{aligned} M(x)f(x) &= C(Id_p - A(x)B(x))f + B(x)f(x) && (1)\\ &= C(f(x) - A(x)g(x)) + g(x) && (2)\\ &= C(f(x) - f(x)) + g(x) = g(x). && (3) \end{aligned}$$

□

Definition 6 (Algebraic Multiplicity[3, Def. 11.9.])
Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a map germ. Define the algebraic multiplicity of f to be $m_A(f) = \dim Q(f)$.

Definition 7 (Geometric Multiplicity[3])
Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a map germ. Define the algebraic multiplicity of f to be $m_G(f)$ equal to the maximal number of preimage points for a representative function on an arbitrarily small neighbourhood.

Definition 8 ($\mathcal{K}'$-equivalence)
Two map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are $\mathcal{K}'$-equivalent if there exists a diffeomorphism Ψ of $(\mathbb{R}^n \times \mathbb{R}^p, (0, 0))$ of the form

$$\begin{aligned} \Psi : \mathbb{R}^n \times \mathbb{R}^p &\to \mathbb{R}^n \times \mathbb{R}^p && (4)\\ (x, y) &\mapsto (\phi(x), \psi(x, y)) && (5) \end{aligned}$$

such that for the graphs of functions f, Γ_f,

1. $\Psi(\Gamma_0) = \Gamma_0$, and
2. $\Psi(\Gamma_f) = \Gamma_g$

More explicitly one may write that for the diffeomorphism $\Psi = (\phi, \psi)$ in the definition it has to hold that,

1. $\psi(x, 0) = 0$ and
2. $g \circ \phi(x) = \psi(x, f(x))$.

Proposition 5 ([3, Prop. 11.10.])
$\mathcal{K}$ and $\mathcal{K}'$ equivalences are identical.

Proof. First the simple direction, if f and g are $\mathcal{K}$-equivalent then they are $\mathcal{K}'$-equivalent.

For the converse direction one applies Hadamard's lemma. Let g and f be $\mathcal{K}'$-equivalent, then there is a diffeomorphism Ψ such that

$$g \circ \phi(x) = \psi(x, f(x)).$$

Each component ψ_j satisfies $\psi_j(x,0) = 0$ for all $j = 1, \ldots, p$ we can apply Hadamard's lemma to write

$$\psi_i(x,y) = \sum_j \chi_{i,j}(x,y) y_j,$$

for $\chi_{i,j} \in \mathcal{E}_n$. It follows for each component that,

$$\psi_i(x, f(x)) = \sum_j \chi_{i,j}(x, f(x)) f_j(x)$$

Define $M(x) := (\chi_{i,j}(x, f(x))$. Then we may write by the definition of ψ

$$g \circ \phi(x) = M(x) f(x)$$

and we conclude that $g \sim f$. □

Differential Geometric Viewpoint

Now follows a short digression to differential geometry. Some details like regularity of maps are omitted, and can found in full length in Lee's book "Smooth manifolds" [2].

Definition 9 (Manifold)
A topological manifold is a second countable Hausdorff space that is locally euclidean.
We call the tuple (U, φ) consisting out of an open subset U of M and a homeomorphism φ from U to an open subset $V \subset \mathbb{R}^n$ a chart. A collection of chart such that the open sets cover M is called an Atlas $\mathcal{A}$.
A topological manifold with an Atlas whose transition maps, i.e. all charts (U_i, φ_i) and (U_j, φ_j) composed to $\tau_{i,j} := \varphi_i \circ \varphi_j^{-1}$, are smooth is called a smooth manifold.

Definition 10 (Fibre Bundle)
Let M, E and F be a topological spaces and $\pi : E \to B$ a map we call the 4-tupel (E, M, π, F) a fibre bundle if π is a continuous surjection, such that for every $x \in M$ exists an open neighborhood $U \subset M$ of x such that there is exists a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & \searrow^{\pi} & \downarrow{\scriptstyle pr_1} \\ & & U \end{array}$$

We call E the total space, M the base space, F the fibre and the collection of all $\{(U_i, \varphi_i)\}$ is called a local trivialization of the bundle.

Definition 11 (Vector Bundle)
Let $(E, M, \pi, \mathbb{R}^k)$ be a fibre bundle. It is called a (real) vector bundle of rank k if the fibers $\mathbb{R}^k$ are vector spaces of dimension k and the trivializations (ϕ, U) induces a linear isomorphism $v \mapsto \phi(x, v)$ between $\pi^{-1}(x)$ and $\mathbb{R}^k$.
Given two trivializations (φ_i, U_i) and (φ_j, U_j) one obtains the composition

$$\begin{aligned}\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^k &\to (U_i \cap U_j) \times \mathbb{R}^k \\ (x, v) &\mapsto (x, g_{i,j}(x)v)\end{aligned}$$

which defines the transition map $g_{i,j} : U_i \cap U_j \to Gl_k(\mathbb{R})$.

Definition 12 (Section of a Bundle)
A section of a fibre bundle is a smooth map $f : M \to E$ such that the base point is preserved, i.e. $\pi(f(x)) = x$ for $x \in M$.

In the setting of smooth manifolds we may define $\mathcal{K}$-equivalence as follows.

Definition 13 ($\mathcal{K}$-equivalence on Manifolds)
Let V be a vector bundle of rank p over M and $f, g \in \Gamma(V)$ section of V. These sections are $\mathcal{K}$-equivalent if there exist choices of charts and local trivializations in which the functions look identical.

Definition 14 ($\mathcal{K}'$-equivalence on Manifolds)
Let E be a fiber bundle whose fibers are p-dimensional Manifolds over M, $f, g \in \Gamma(E)$. These sections are $\mathcal{K}'$-equivalent if there exists choices of charts and local trivializations such that the local expressions coincide.

References

[1] V. I. Arnold, A. N. Varchenko, and S. M. Gusein-Zade. *Singularities of differentiable maps: Volume II Monodromy and asymptotic integrals*, volume 83. Springer Science & Business Media, 2012.

[2] J. M. Lee. *Smooth manifolds.* Springer, 2012.

[3] J. Montaldi. *Singularities, bifurcations and catastrophes.* Cambridge University Press, 2021.