Finite determinacy for contact equivalence

Following Montaldi's book

Plan

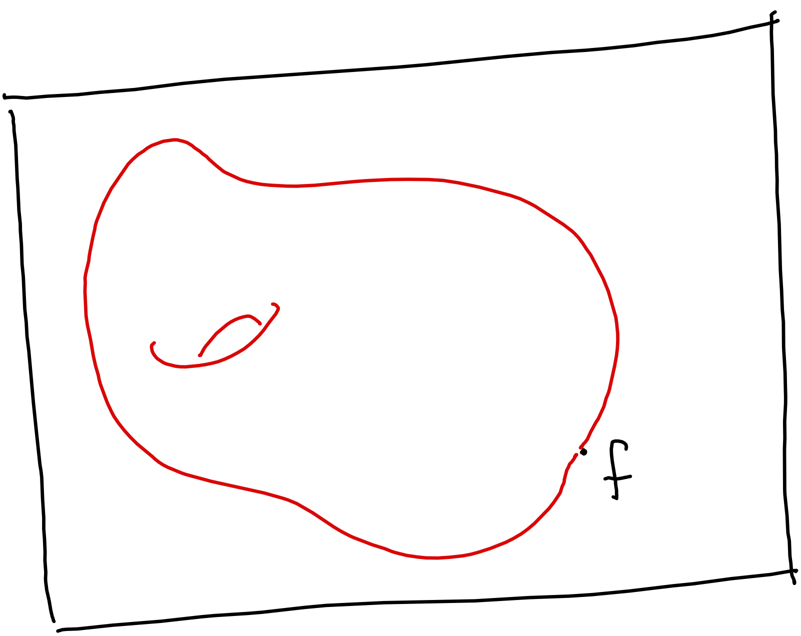
1. DREAM
2. Recall: Contact equivalence
3. Thom-Levine principle
4. Constant tangent spaces
5. Finite determinacy
6. Invariants

1. DREAM

$f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ germ of a map

$\sim_G$ equivalence relation on map germs

$\mathcal{E}_n^p$ vector space



Equiv. class of f

How to classify?

- Good representatives ← The main goal of finite determinacy
- Invariants

What are the reasonable germs to classify?

If $\sim_g$ is induced by a group action (as in the case of contact equivalence):

One can form the quotient space

$$X/G$$

Our quotient space locally looks like a manifold iff

The tangent space of $\mathcal{E}_n^p$ at f

$$\operatorname{Codim}(f,\mathcal{G}) = \dim\left(\Theta(f) \big/ T\mathcal{G}\cdot f \right) < \infty$$

The tangent space of $\mathcal{G}\cdot f$ at f

Thus we will only consider finite codimensional germs!

Thm

TFAE for any reasonable $\sim_{\mathcal{G}}$

today we will prove this part

(i) f is finitely determined w.r.t $\mathcal{G}$

$\Uparrow$

(ii) f has finite codimension

(iii) f possesses versal unfoldings

these correspond to a cobasis

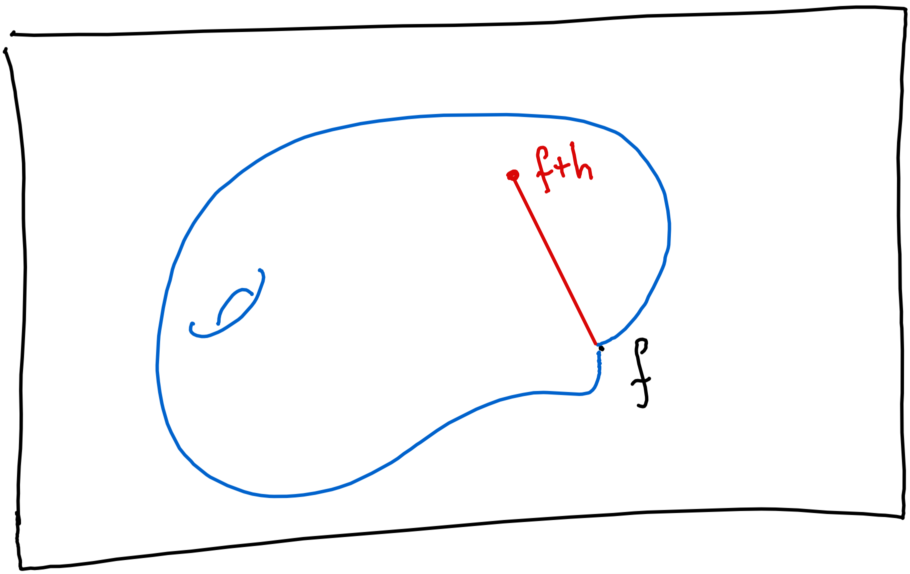
Def

A map germ f is k-determined if $\forall\ h \in m_n^{k+1} \mathcal{E}_n^p$, $f+h \sim_{\mathcal{G}} f$

i.e. the $\mathcal{G}$- equiv. class is determined by the k-jet.

f is finitely determined if $\exists\, k$ s.t. f is k-det.

How would you prove $2 \Rightarrow 1$?



$$f+h \overset{?}{\underset{G}{\sim}} f \quad \text{i.e. } f+h \in G\cdot f$$

DREAM

$$\forall s \in [0,1] \quad f_s = f + sh \in G\cdot f$$

We would need:

$$\dot{f}_s \in TG\cdot f_s \quad \forall s \quad \text{(smoothly)}$$

In this special case:

$\dot{f}_s = h$ "constant"

How to make this precise?

II. Recall: Contact equivalence

$$f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$$

<u>Def</u>

$f \sim_K g$ if $\exists\, \phi, M$:

$$\begin{tikzcd}
(\mathbb{R}^n, 0) \arrow[r, "g"] \arrow[d, "\phi"', "\text{diffeo}"'] & (\mathbb{R}^p, 0) \arrow[d, "M \in GL_p(\mathcal{E}_n)"] \\
(\mathbb{R}^n, 0) \arrow[r, "f"] & (\mathbb{R}^p, 0)
\end{tikzcd}$$

i.e. $f \circ \phi(x) = M(x) \cdot g(x)$

Prop

$$f \sim_K g \Rightarrow f^{-1}(0) \underset{\text{diffeo}}{\cong} g^{-1}(0)$$

Contact tangent space

$$TK \cdot f = tf(m_n \theta_n) + I_f \theta(f)$$

$$T_e K \cdot f = tf(\theta_n) + I_f \theta(f)$$

$$\operatorname{codim}(f, K) = \dim \left(\theta(f) / T_e K \cdot f \right)$$

Rem

$$\dim \left(\theta(f) / T_e K \cdot f \right) < \infty \Leftrightarrow \dim \left(\theta(f) / TK \cdot f \right) < \infty$$

Algebraic Lemma

f has finite codim w.r.t. $\mathcal{K}$

$$\iff$$

$\exists r \quad m_n^r\, \theta(f) \subset T\mathcal{K}\cdot f$

Prop

$$f \sim_{\mathcal{K}} g \implies \operatorname{codim}(f, \mathcal{K}) = \operatorname{codim}(g, \mathcal{K})$$

Very intuitive from the geometric picture!

3. Thom - Levine principle

Let f_s be a smooth 1-param family of germs in $\mathcal{E}_n^p$.

Then f_s is a $\mathcal{K}$-trivial family

(i.e. $\exists$ smooth family ϕ_s, M_s : $f_s \circ \phi_s = M_s f_0$)

$$\Longleftrightarrow$$

$\dot{f}_s \in T\mathcal{K} \cdot f_s$ smoothly

$$\dot{f}_s(x) = d(f_s)_x u(x,s) + G_s(x) f_s(x)$$

Proof

$\boxed{\Rightarrow}$ This is how we calculated $TK \cdot f_s$!

$\boxed{\Leftarrow}$

$$\dot{f}_s(x) = d(f_s)_x \, u(x,s) + G_s(x) \, f_s(x)$$

Want: $$f_s \circ \phi_s(x) = M_s(x) \cdot f_0(x)$$

Equivalently: $$M_s^{-1}(x) \, f_s \circ \phi_s(x) = f_0(x)$$

So if the LHS does not depend on s we get a k-trivial family!

$$\frac{d}{ds}\left(M_s^{-1}(x) \, f_s \circ \phi_s(x) \right) \stackrel{?}{=} 0$$

$$\frac{d}{ds}\left(M_s^{-1}(x)\; f_s \circ \phi_s(x)\right) =$$

$$\left(\dot{M}_s^{-1}(x)\right) f_s \circ \phi_s(x) + M_s^{-1}(x)\left(\underbrace{\dot{f}_s \circ \phi_s(x)}_{= \, d(f_s)_{\phi_s(x)}\, u(\phi_s(x),s) + G_s(\phi_s(x))\, f_s \circ \phi_s(x)} + d(f_s)_{\phi_s(x)} \cdot \frac{d}{ds}\phi_s(x)\right)$$

Rearranging the terms, we get:

$$M_s^{-1}(x)\; d(f_s)_{\phi_s(x)}\left(\underbrace{\frac{d}{ds}\phi_s(x) + u(\phi_s(x),s)}_{\text{choose } u \text{ so that this is } 0}\right)$$

choose u so that this is 0
i.e. let ϕ_s be the flow corresponding to $-u$!

$$+\left[\underbrace{\left(\dot{M}_s^{-1}(x)\right) + M_s^{-1}(x)\; G_s(\phi_s(x))}\right] f_s(\phi_s(x))$$

Solve this ODE to make this part 0 aswell

$\square$

4. Constant tangent spaces

Q

How to prove smoothness in the Thom-Levine principle?

Very useful case:

The tangent space to f_s is "constant"

Q

How to make this precise?

$$\mathcal{E}_{n,I} := \left\{ f : \mathbb{R}^n \times I \longrightarrow \mathbb{R} \text{ germs along } \{0\} \times I \right\}$$

$$m_{n,I} := \{ \text{germs vanishing on } \{0\} \times I \}$$

$$\Theta_{n,I} := \mathcal{E}_{n,I}\, \Theta_n$$

<u>Def</u> (relative tangent space)

$\{f_s\}_{s \in I} \in \mathcal{E}^p_{n,I}$ family of germs,

$$T_{rel} K \cdot f_s := \mathcal{E}_{n,I}\, T K \cdot f_s$$

<u>Rem</u>

$T_{rel} K \cdot f_s$ is an $\mathcal{E}_{n,I}$-module

How to get back $Tk \cdot f_{s_0}$?

$$\begin{aligned} ev: \mathcal{E}_{n,I} &\longrightarrow \mathcal{E}_n \\ h &\longmapsto h\big|_{s=s_0} \end{aligned}$$

is a homomorphism of $\mathcal{E}_n$-modules that can be extended to $\mathcal{E}_{n,I}$ modules

Moreover,

$$ev\left(T_{rel}k \cdot f_s\right) = Tk \cdot f_{s_0}$$

Def

Let $\{f_s\}_{s\in I} \in \mathcal{E}_{n,I}^p$ be a smooth family.

The tangent space is constants if

$\exists$ vector fields $v_1, \dots, v_r \in \Theta(f) \cong \mathcal{E}_n^p$

s.t. $T_{rel} K \cdot f_s = \mathcal{E}_{n,I}\{v_1, \dots, v_r\}$.

Thm *

Suppose $f_s = f + sh$ is a family of germs. Let M be a finitely generated $\mathcal{E}_n$-submodule of $\Theta(f)$.

Suppose

i) $h \in M$

ii) $\mathcal{E}_{n,I} M \subset T_{rel}\,\mathcal{K} \cdot f_s$

Then $\{f_s\}_{s\in I}$ is $\mathcal{K}$-trivial.

<u>Proof</u>

The claim follows from the Thom-Levine principle as

$$\dot{f}_s = h \in T\mathrm{rel}\, K \cdot f_s \quad \forall s$$

by the hypothesis.

We only have to check that

$$\dot{f}_s \in T\mathrm{rel}\, K \cdot f_s \text{ smoothly.}$$

Since $h \in M$ $\exists\, \beta_j$ s.t.

generators of M

$$h(x) = \sum \beta_j(x) v_j(x)$$

This expression does not depend on s thus it is smooth

$\square$

5. Finite determinacy

Thm

f has finite codimension

$\Rightarrow$ f is finitely determined

Proof

f finite codim $\xrightarrow{\text{Alg. lemma}}$ $\exists k \quad m_n^k \theta(f) \subset TK \cdot f$

Let $h \in m_n^{k+1} \mathcal{E}_n^p$, $f_t := f + th$.

Lemma

$T_{rel}\, K \cdot f_s$ is constant!

Proof of the lemma

Since $m_n T_{rel} K \cdot f$ is constant

we have to show: $m_{n,I} T_{rel} K \cdot f_s = m_{n,I} T_{rel} K \cdot f$.

1. $\subseteq$

$$m_{n,I} T_{rel} K \cdot f_s = m_{n,I} T_{rel} K \left(f + sh\right)$$

$$\subseteq m_{n,I} T_{rel} K \cdot f + s\, m_{n,I} T_{rel} K \cdot h$$

$h \in m_n^{t+1} \mathcal{E}_n^p$

$\Downarrow$

$TK \cdot h \subseteq m_n^t \mathcal{E}_n^p$

$$\subseteq m_{n,I} T_{rel} K \cdot f + s\, m_{n,I}^{t+1} \mathcal{E}_n^p$$

$m_n^t \Theta(f) \subset TK \cdot f$

$$= m_{n,I} T_{rel} K \cdot f$$

2. $\supseteq$

$$f = f_s - sh$$

$$\begin{aligned}
m_{n,I}\, T_{rel}\, K \cdot f &= m_{n,I}\, T_{rel} \left(f_s - sh\right) \\
&\subseteq m_{n,I}\, T_{rel}\, K \cdot f_s + s\, m_{n,I}\, T_{rel}\, K \cdot h \\
&\subseteq m_{n,I}\, T_{rel}\, K \cdot f_s + s\, m_{n,I}^{k+1}\, \mathcal{E}_n^p \\
&\subseteq m_{n,I}\, T_{rel}\, K \cdot f_s + m_{n,I}\, T_{rel}\, K \cdot f
\end{aligned}$$

Nakayama's lemma $\Longrightarrow$ $m_{n,I}\, T_{rel}\, K \cdot f \subseteq m_{n,I}\, T_{rel}\, K \cdot f_s$

$\boxed{\text{Lemma}}$

Thus $m_{n,I}\, T_{rel}\, K \cdot f_s$ is constant and contains h $\overset{*}{\Longrightarrow}$ f_s is a K-trivial family

$\Longrightarrow$ $f = f_0 \sim f_1 = f + h$

$\boxed{\text{Thm}}$

6. Invariants

Def

The local algebra of f

is $\quad Q(f) := \mathcal{E}_n / I_f$

Thm

f, g finitely k-det $\Rightarrow$

$$f \sim_K g \iff Q(f) \cong Q(g)$$

Algebraic multiplicity

$$m_A(f) := \dim Q(f)$$

Geometric multiplicity

$$m_G(f) = \max_{x \in B_\varepsilon(0)} |f^{-1}(x)|$$

Thm

If f is finitely K-det

$$\Rightarrow \quad m_G(f) \leq m_A(f)$$

Moreover, for complex analytic maps

$$m_G(f) = m_A(f).$$

The contact codimension

$\operatorname{codim}(K, f)$ is an invariant

of the variety $(f^{-1}(0), I_f)$

One can show that

$$\operatorname{codim}(K, f) \leq \operatorname{codim}(R, f)$$

$$\begin{array}{ccc} \| & & \| \\ \tau(f) & & \mu(f) \end{array}$$

Tjurina number

Milnor number