# Introduction to Bifurcation Theory

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## §1. What is a Bifurcation problem?

A Bifurcation problem is in its most general[1] form an equation:

$$G(x;\lambda) = 0$$

for a (smooth)[2] function $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ with state variable $x \in \mathbb{R}^n$ and parameter $\lambda \in \mathbb{R}$. We will write $g_\lambda(x) := G(x;\lambda)$, letting us denote the solutions for a specific $\lambda$ as $g_\lambda^{-1}(\{0\})$. In case of $x \in \mathbb{R}$ it is often useful to graph the solutions of a bifurcation problem, with $\lambda$ on the horizontal and $x$ on the vertical axis. We will call such a diagram the **bifurcation diagram** of a bifurcation.

> **Example** (The Saddle-node Bifurcation).
> The most basic example of a bifurcation is $G(x;\lambda) = x^2 - \lambda$ with $x \in \mathbb{R}$. As seen in Figure 1, moving $\lambda$ from negative to positive one starts with having two solutions, which "merge" at $\lambda = 0$ and vanish for $\lambda > 0$.

> **Example** (The Pitchfork Bifurcation).
> Getting a bit more complicated, the pitchfork bifurcation is $H(x;\lambda) = x^3 + \lambda x$, producing a bifurcation diagram that unsurprisingly looks like a pitchfork (see Figure 2). In this case one starts with 3 solutions for $\lambda < 0$, which merge to one for $\lambda \geq 0$.

[1] We could allow even more general functions from $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, but in practice this is done rarely and it would complicate this introduction without providing any substantial advantages.

[2] All functions are presumed to be smooth.

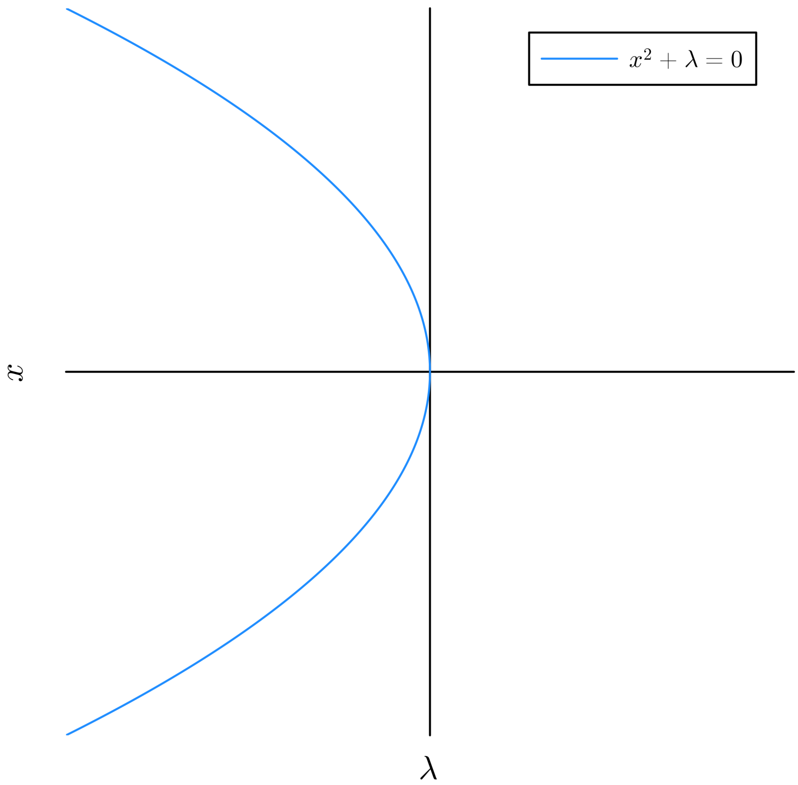



Figure 1: The bifurcation diagram of the Saddle-node Bifurcation

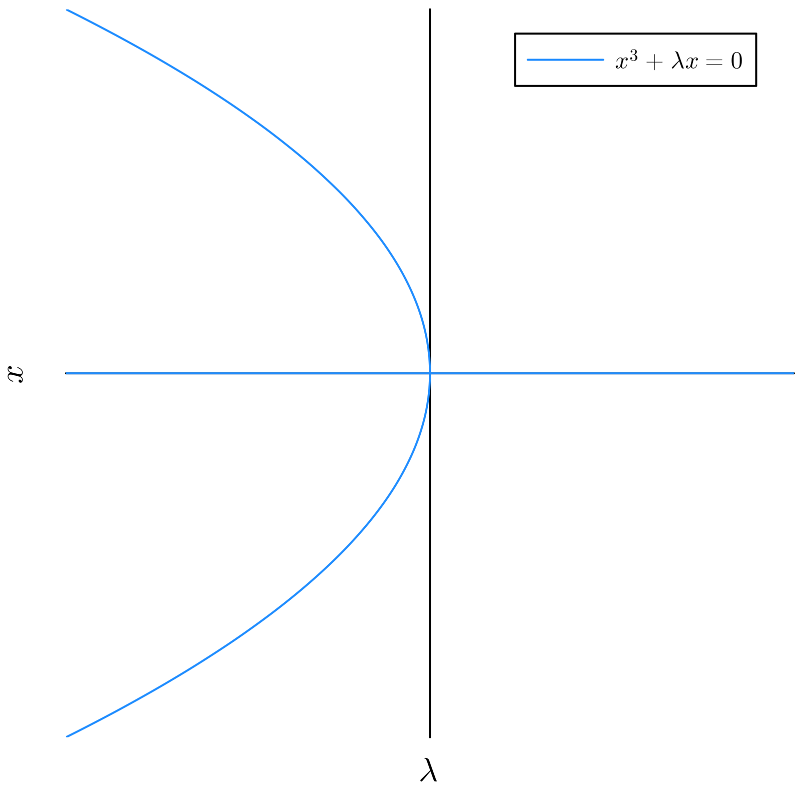



Figure 2: The bifurcation diagram of the Pitchfork Bifurcation

Bifurcation theory is of course not (just) about drawing the bifurcation diagrams of interesting bifurcations. We want to actually gain a better understanding of these bifurcations. A natural question to start with is the stability of these bifurcation, i.e. given a bifurcation problem $G(x;\lambda) = 0$, how similar is it to the problem $G(x;\lambda) = \varepsilon$ for a small $\varepsilon > 0$.

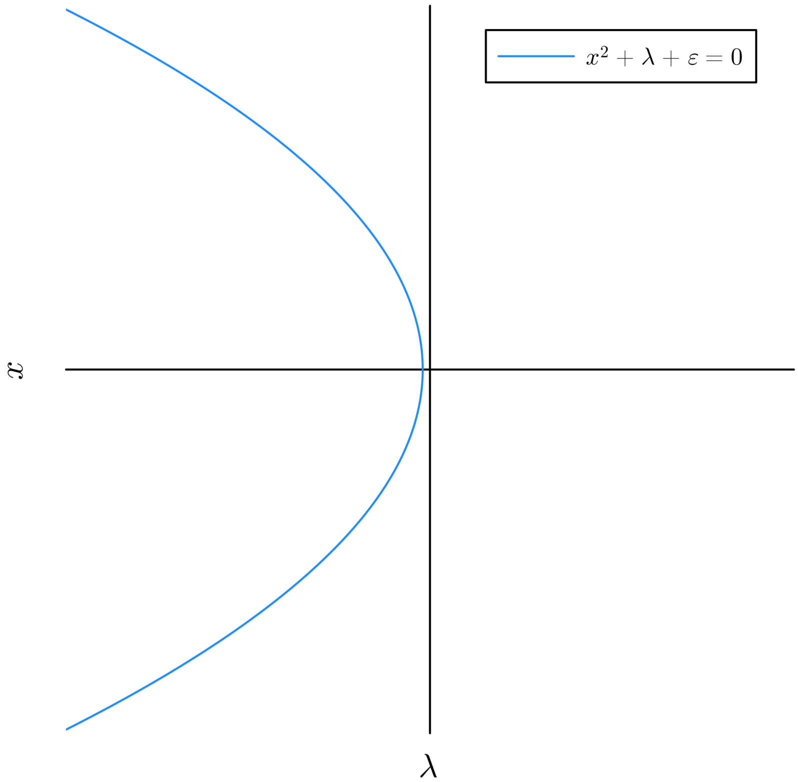



Figure 3: The perturbed Saddle-node Bifurcation

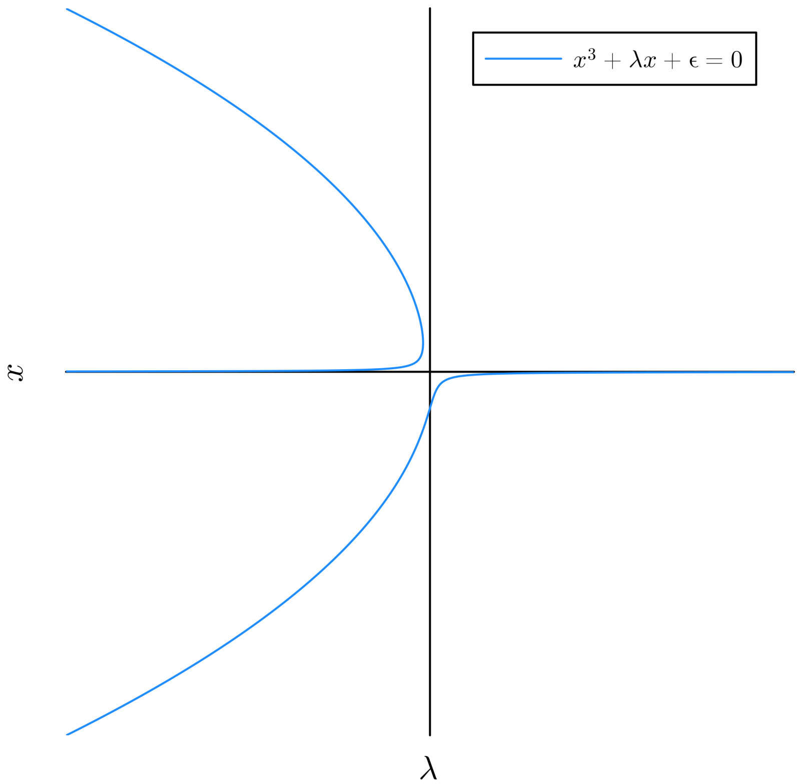



Figure 4: The perturbed no-longer-a-pitchfork Bifurcation

The result of small perturbations of the Saddle-node and Pitchfork Bifurcation can be seen in Figure 3 & Figure 4: The Saddle-node Bifurcation does not change “qualitatively”, but the Pitchfork-Bifurcation does, it can no longer be called a Pitchfork-Bifurcation in any sense of that word. We can’t yet explain why this happens, so let’s try doing that.

## §2. The Path Approach

Let us limit ourselves to the pitchfork bifurcation, as it was the one who did actually change. Trying to put this into more familiar terms, let’s look at all of the constant perturbations of the pitchfork bifurcation at the same time, by defining:

$$F(x;\lambda;\varepsilon) = x^3 + \lambda x + \varepsilon$$

Explaining the behavior of the pitchfork bifurcation under small perturbations is now equal to explaining how small changes in $\varepsilon$ change how $F(x;\lambda;\varepsilon) = 0$ behaves under small changes of $\lambda$.

What makes this different from the problems we considered before is the “two-layered-ness”, having to consider the changes of $\lambda$ and $\varepsilon$ as separate things. The easier question would be how $F(x;\lambda;\varepsilon)$ behaves under small perturbations of $\lambda$ and $\varepsilon$ at the same time. Let us try to answer this simpler question, maybe this helps us answering the more complicated one.

For that we can use the tools already at our disposal. Let us define:

$$G(x;u,v) = x^3 + ux + v$$

We already know this as the versal unfolding of $x^3$. Mapping the set of solutions, i.e.

$$Z_G := \{\,(x,u,v) \in \mathbb{R}^3 \mid G(x;u,v) = 0\,\}$$

gives us the familiar picture of the cusp catastrophe seen in Figure 5. The best way to look at the behavior of this unfolding under permutations is to look at its singularity set, that is

$$\Sigma_G := \{\,(x,u,v) \in Z_G \mid \partial_x G = 0\,\}$$

but as usual, this is a bit too much information, so we will just look at the discriminant, i.e.

$$\Delta_G := \pi(\Sigma_G)$$

with $\pi$ being the projection of $(x,u,v)$ to $(u,v)$, giving us the cusp as seen in Figure 6. This is quite useful, since this is a problem we already understand quite well, but how do we now add the “two-layered-ness” of our bifurcation problem? To achieve this let us first connect the unperturbed pitchfork bifurcation to this diagram, which we will denote by $P$. The unperturbed pitchfork bifurcation can also be viewed as an unfolding of $p_0 = x^3$ and since $G$ is a $\mathcal{K}$-versal unfolding of exactly this function, we know that there is an $h$ that induces $P$ from $G$, that is $P = h^*G$. In this case the definition of $h$ can be seen directly from the definition of $G$ and $P$:

$$P(x;\lambda) = (h^*G)(x;\lambda) = G(x;h(\lambda)) = G(x;\lambda,0)$$

hence $h(\lambda) = (\lambda, 0)$, similarly for the perturbed version we define $h_\varepsilon(\lambda) = (\lambda,\varepsilon)$, leading to $F = h_\varepsilon^* G$.

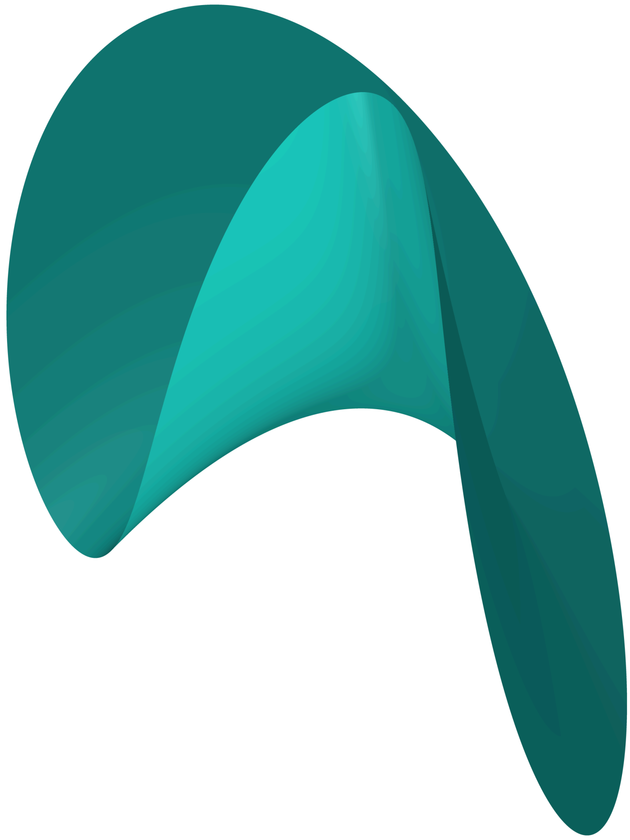


Figure 5: The cusp catastrophe

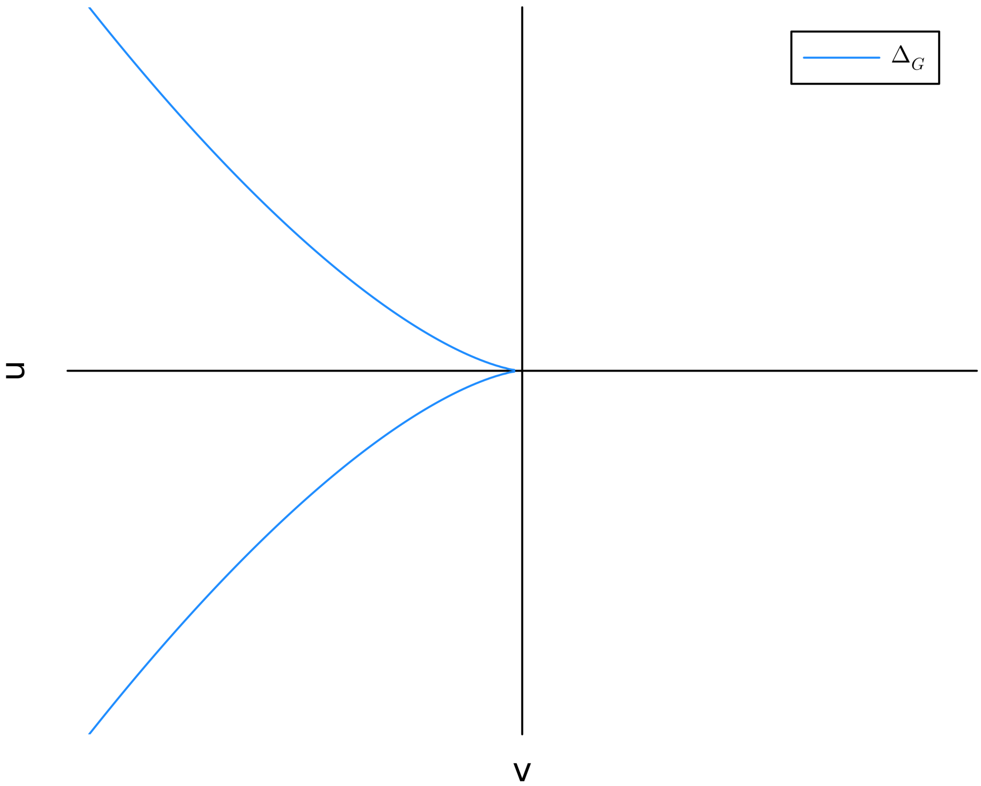



Figure 6: The familiar cusp

As smooth maps from $R$ to $R^2$ these can now be considered as paths in $\mathbb{R}^2$, with $\mathbb{R}^2$ being the parameter space of $G$ and therefore the same space in which $\Delta_G$ lives. Looking at their image as seen in Figure 7, one sees directly (unsurprisingly) that the bifurcation point of the pitchfork bifurcation corresponds to the intersection of $h$ with $\Delta_G$. Since we are interested in the change that happened to the pitchfork bifurcation after the small constant perturbations a natural question is now in which way $h$ and $h_\varepsilon$ differ from each other/which properties $h$ has that are “unstable” in the sense that any perturbation of it won’t have this property anymore (in general). This leads us to two observations:

1. $h$ is tangent[3] to $\Delta_G$, while $h_\varepsilon$ isn’t
2. $h$ crosses $\Delta_G$ once, so does $h_\varepsilon$. This interestingly is a stable property for $h_\varepsilon$, but not for $h$, general perturbations also include rotations and any rotation (without translation) of $h$ makes it cross $\Delta_G$ twice.

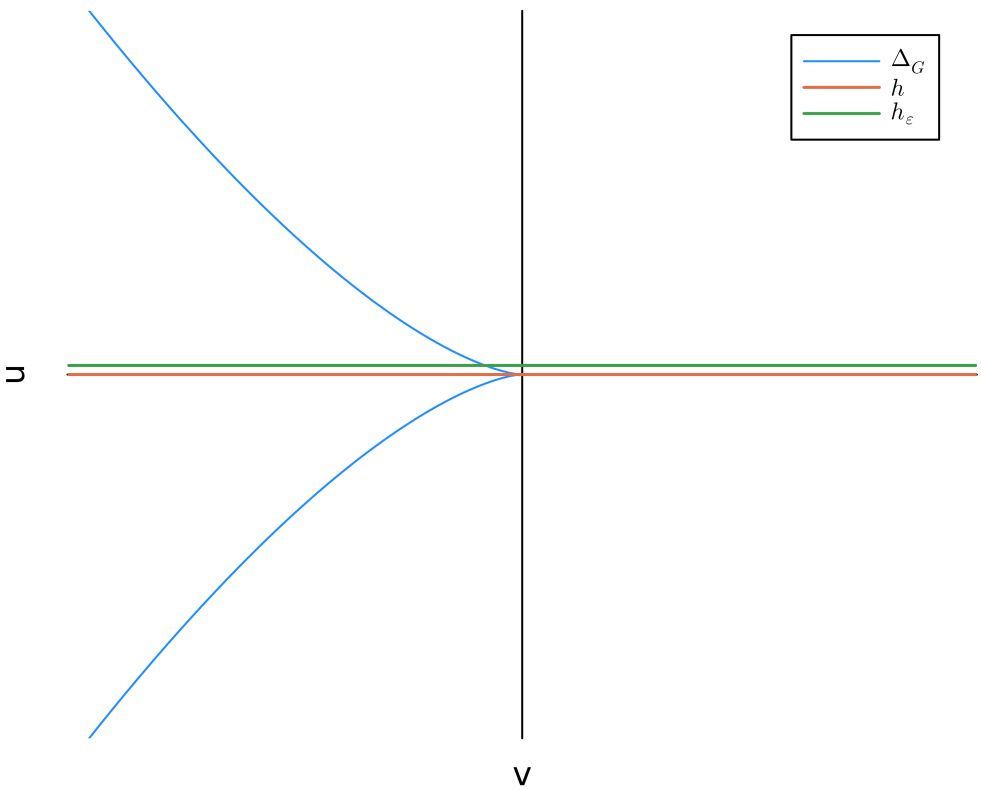



Figure 7: $\Delta_G$ with the two paths that induce the pitchfork bifurcation and a small constant perturbation of it

Based on these observation one would hope that any perturbation of the pitchfork bifurcation can be represented by a perturbation of this path and vice versa. Before making this statement more rigorous, let us experiment some more. By including rotations we should have total control about the relevant properties of $h$, so let us define in general:

$$h_{a,b}(\lambda) = (\lambda, a\lambda + b)$$

In Figure 8 one sees all interesting perturbations of $h$ with them being all qualitatively different from each other:

[3]The notion of being tangent is in this case a bit more complicated since the discriminant is not a smooth manifold in the origin, but there is a hopefully intuitive way to see that this can be treated like a tangent, but more on that later

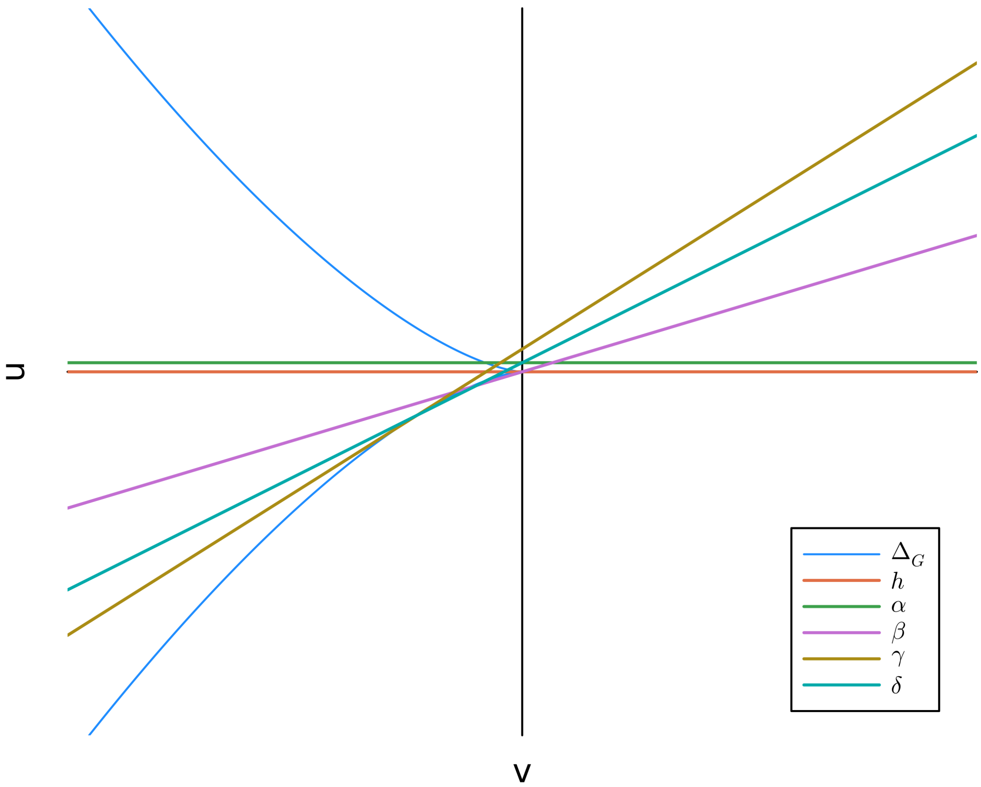


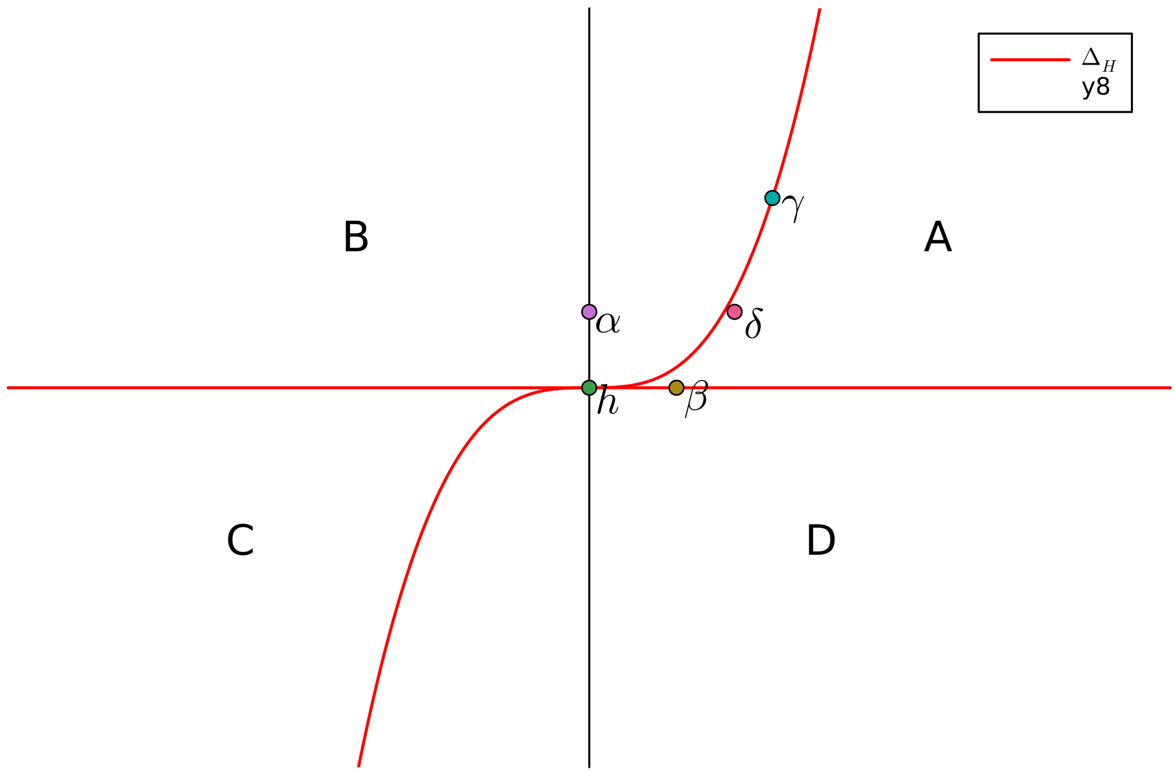

Figure 8: $\Delta_G$ with all interesting perturbations of $h$

- $h$ is of course the only path being tangent to $\Delta_G$ and crossing it in only one point, which corresponds to the bifurcation point in the original pitchfork bifurcation as seen in Figure 2
- $\alpha$ still crosses $\Delta_G$ once, but does but is not tangent to it. Note that in this case crossing $\Delta_G$ once is a stable property and hence, since not being tangent is also stable, its induced bifurcation should be stable and have one bifurcation point, this can be seen in Figure 10
- $\beta$ crosses $\Delta_G$ twice and is not tangent to it. In contrast to $\alpha$ the number of intersections is not stable, since small perturbations either cross $\Delta_G$ once (like $\alpha$) or thrice (like $\delta$). Hence the induced bifurcation should have two bifurcation points and be not stable, with small perturbations producing either the bifurcation induced by $\alpha$ or the one induced by $\delta$, this can be seen in Figure 11
- $\gamma$ crosses $\Delta_G$ twice and is tangent to it. Small perturbations of it are no longer tangent and cross $\Delta_G$ either once or thrice, so the induced bifurcation should have two bifurcation points, with one of them being unstable, with perturbations either erasing it or turning it into two bifurcation points, this can be seen in Figure 12
- $\delta$ crosses $\Delta_G$ thrice without being tangent to it, so its bifurcation diagram should have three bifurcation points, all of them stable, this can be seen in Figure 13

This is already quite useful. By just looking at the perturbed paths we can extract quite a lot of information. Or at least we hope so, since all of this is just a heuristic approach right now. But there is a way to make this a bit more rigorous – how the induced bifurcation behaves under these paths is equivalent to asking how the bifurcation

$$H(x;\lambda;a,b) = x^3 + \lambda x + ax + b$$

behaves under perturbations of its parameters $a$ and $b$ ($\lambda$ is taken as just a normal variable here). This is a familiar question we can answer by graphing its determinant together with the values corresponding to the paths as seen in Figure 9




One immediately sees that the perturbed paths that are themselves unstable correspond to the points in the parameter space on the determinant, “proving” that in this case unstable paths do correspond to unstable bifurcations. All the bifurcations in the different areas should also be qualitatively similar to each other e.g. all bifurcations in area A should look like the one induced by $\delta$.

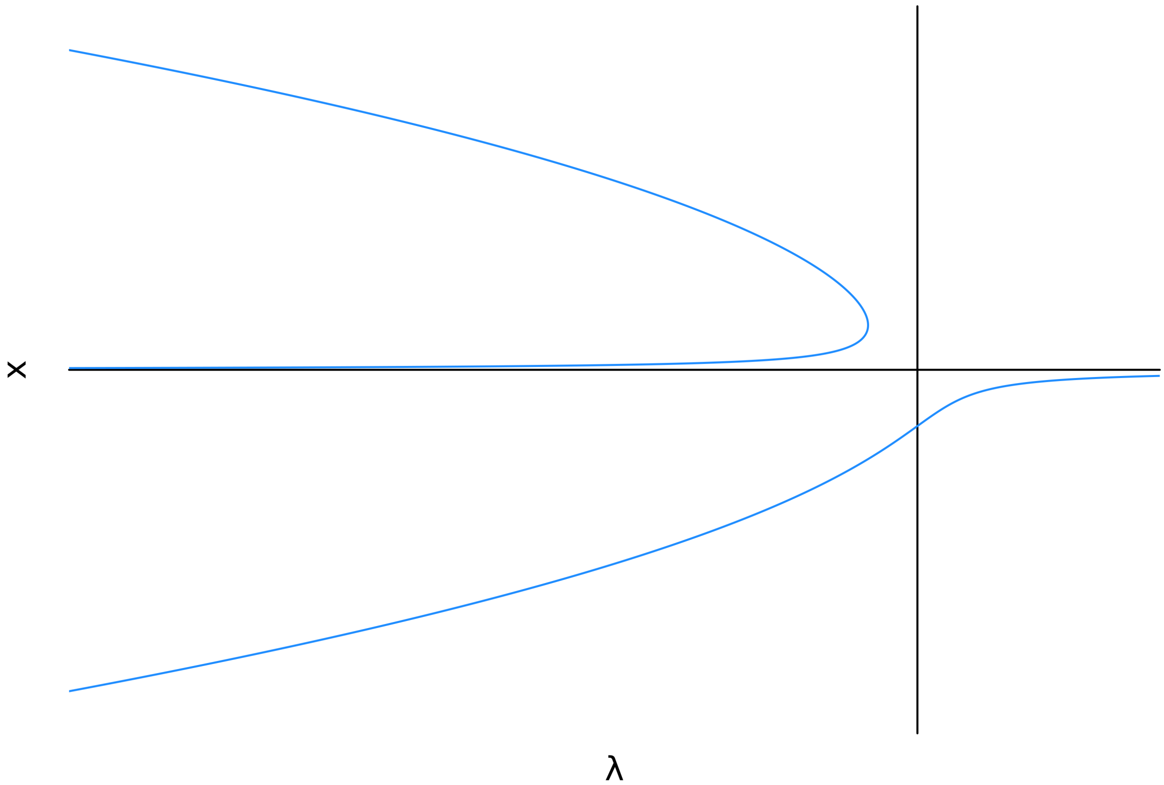



Figure 10: The Bifurcation induced by $\alpha$, the upper part of the graph consists of a saddle-node bifurcation, the lower part does not have a bifurcation point.

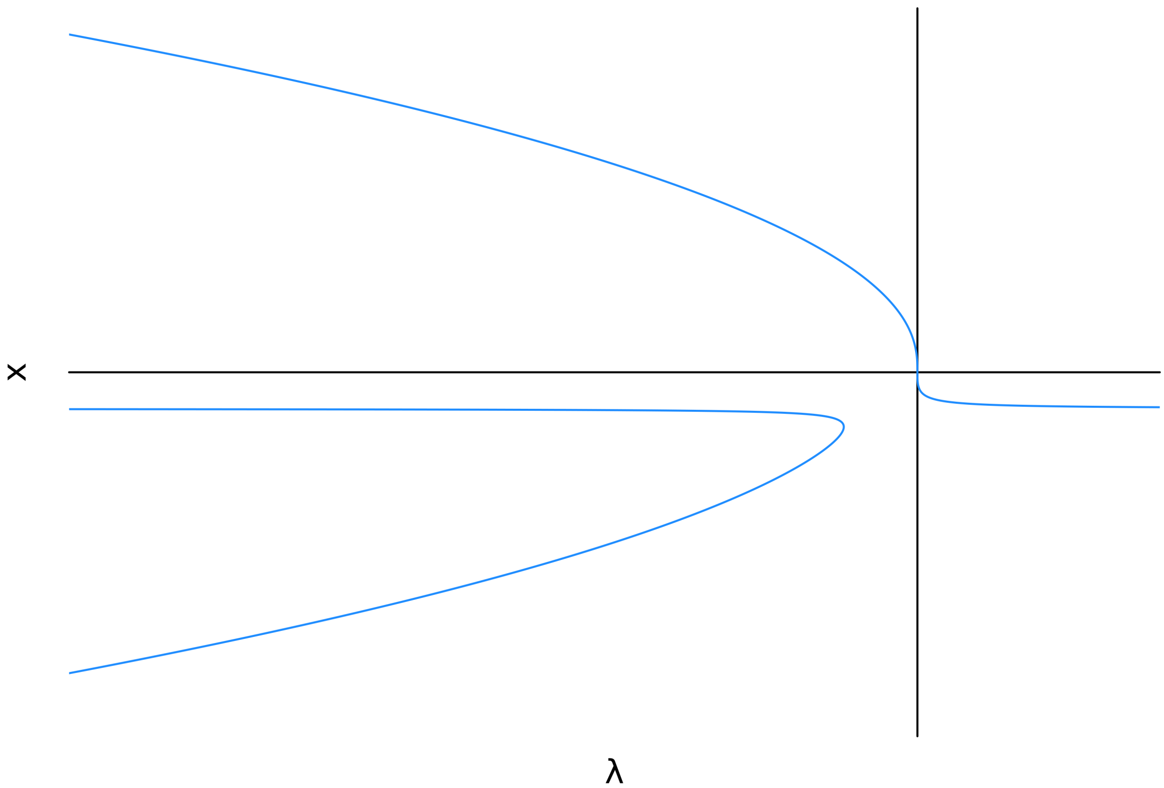



Figure 11: The Bifurcation induced by $\beta$, the lower part consisting of a saddle-node bifurcation and the upper is a so called hysteresis bifurcation, which is itself unstable, since it can either split in two or vanish.

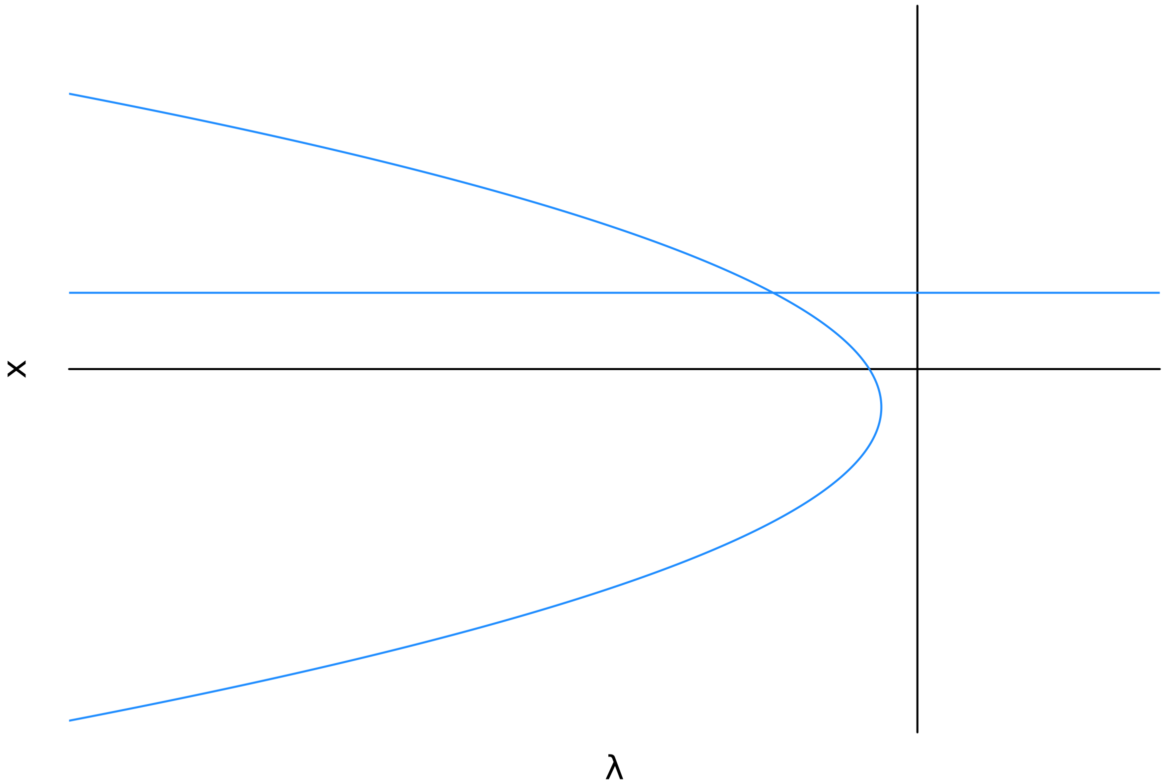



Figure 12: The Bifurcation induced by $\gamma$, the crossing is called a transcritical bifurcation and the point at the right is a saddle-node bifurcation

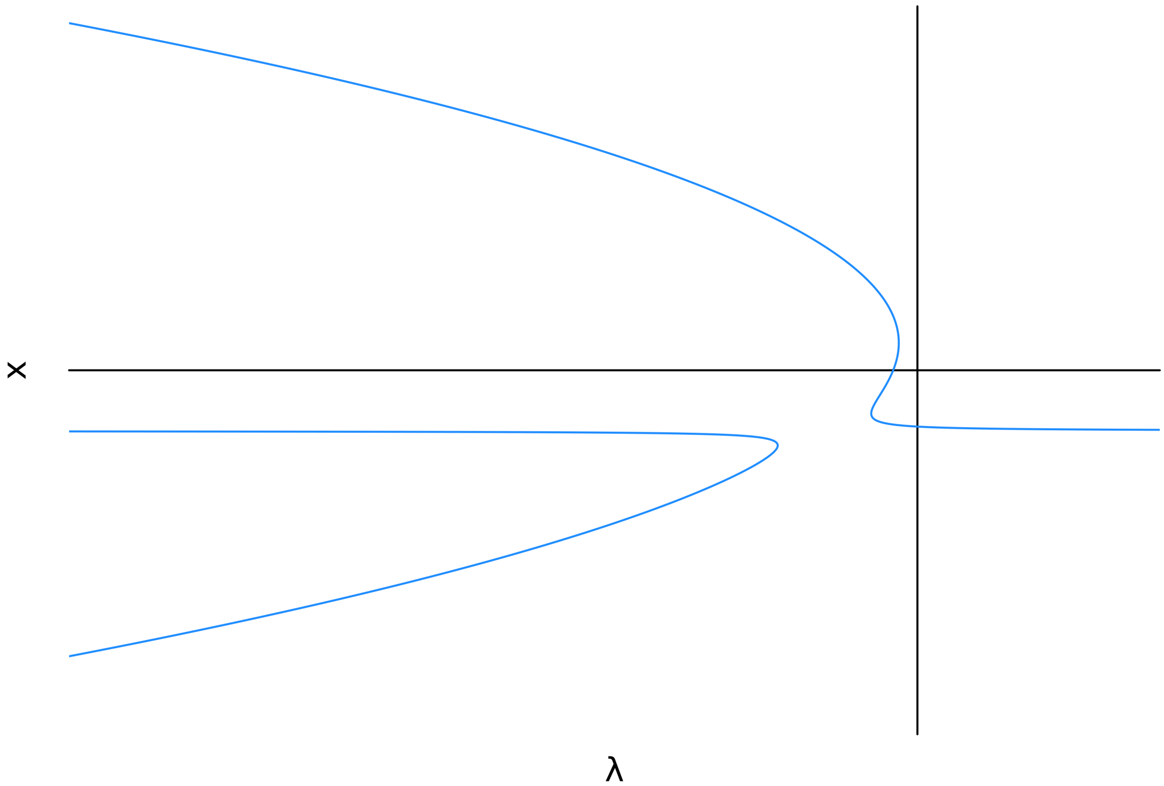



Figure 13: The Bifurcation induced by $\delta$, consisting of three saddle-node bifurcation

All of this confirms our previous work, which is good. A natural question to ask which we did not answer yet is whether these are all the bifurcations or whether we missed any. While we can't answer this with certainty yet, one indicator that these are indeed all the possible perturbations is that the unfolding we chose is versal i.e. every other unfolding (and every perturbation is also an unfolding) of the pitchfork can be induced from it and this will indeed turn out to be true. In Summary our heuristic approach, given a bifurcation $G(x;\lambda)$ is:

1. Find the versal unfolding $H(x;\lambda;u)$ of $g_0$ and the map $h$ such that $G$ is $\mathcal{K}$-equivalent to $h^*H$
2. Figure out how small perturbations of $h$ change "the way $h$ meets $\Delta_G$" and find the simplest path parametrization representing every relevant perturbation
3. Us this path parametrization to find a versal unfolding of $G$ and/or classify all possible perturbation results

# §3. A Little Sprinkle of Rigor

While all of this has hopefully been a illuminating it also leaves a lot to be desired since a lot of statements were just based on hope and a lot of definitions just appealed to intuition. While doing all of this completely rigorous is not feasible in this text, we at least want to lay some groundwork to convince the reader that we are not just guessing and hoping.

## §3.1. Variety is the spice of life

Talking about paths being tangent to the discriminant made intuitive sense so far, but had the major hurdle that the notion of tangency as we know it only works for smooth manifolds, but in general the discriminant isn't one. The solution to this is the algebraic geometric concept of a Variety, or to be more exact a semi-algebraic Variety. Then one can define the logarithmic tangent space of the Variety, which matches the usual definition of tangent space on all the points where the Variety is also a smooth manifold. This allows to not only talk about the tangency of a single path, but also about the module of vector field germs tangent to the Variety, allowing us to properly handle discriminants. Doing all of this in proper detail would take too much time. See Chapter 19 of [1] for a good introduction, we will just take the concept of the discriminant being a Variety and having a tangent space as granted.

## §3.2. "How $h$ meets $\Delta_G$"

Another thing we swept under the rug was how one would "qualitatively" describe how a path $h$ meets a discriminant. Luckily we already answered a similar question in the past, the way how a path (or any function) meets the origin is captured by $\mathcal{K}$-equivalence. Extending this to a Variety $V$, we define:

**Definition 3.2.1** ($\mathcal{K}_V$-equivalence).
Let $h, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two map germs and $V \subseteq \mathbb{R}^p$ a Variety. We call $h$ and $g$ **$\mathcal{K}_V$-equivalent**, denoted by $h \sim_{\mathcal{K}_V} g$, if there is a diffeomorphism germ $\Psi : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ of the form

$$\Psi(x, y) = (\phi(x), \iota(x, y))$$

such that

1. $\Psi(\mathbb{R}^n \times V) = \mathbb{R}^n \times V$
2. $\Psi(\Gamma_f) = \Gamma_g$

with $\Gamma_f$ being the graph of $f$.

These conditions are equivalent to requiring that $\psi(x, y) \in V$ for every $y \in V, x \in \mathbb{R}^n$ and $(g \circ \phi)(x) = \psi(x, f(x))$, especially implying that $f(x) \in V$ if and only if $(g \circ \phi)(x) \in V$.

For our previously considered equivalence relations the concept of their respective tangent spaces was a quite useful. To define the tangent space for this relation we first define $\theta_V$ to be the $\mathcal{E}_p$-module generated by these polynomial vector fields in $\mathbb{R}^p$ that are tangent to $V$ and $\theta_{V,0} = \theta_V \cap \mathfrak{m}_p \theta_p$ as these that also vanish at the origin. By the same rationale as in the regular $\mathcal{K}$-equivalence case this leads to

**Definition 3.2.2.**
Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^p, 0)$ be a smooth map germ and $V \subseteq \mathbb{R}^p$ an algebraic Variety. The **$\mathcal{K}_V$-tangent space** of $f$ is defined to be

$$T\mathcal{K}_V \cdot f = \mathrm{t}f(\mathfrak{m}\theta) + f^*\theta_{V,0}$$

The **extended** tangent space is defined to be

$$T_e\mathcal{K}_V \cdot f = \mathrm{t}f(\theta) + f^*\theta_V$$

with $f^*\theta_V$ being the $\mathcal{E}$-module of vector fields along $f$ that are tangent to $V$, similar for $f^*\theta_{V,0}$

Most of the useful theorems we proved for $\mathcal{K}$-equivalence hold by an analogous proof (which is hence left to the reader), with the most important probably being the Thom-Levine theorem and the implication of finite determinacy by finite $\mathcal{K}_V$-codimension.

**Theorem 3.2.1** (The Thom-Levine theorem for $\mathcal{K}_V$-equivalence)**.**
Let $f_s$ be a smooth family of germs in $\mathcal{E}^p$ and $V \subseteq \mathbb{R}^p$ a variety. $f_s$ is a $\mathcal{K}_V$-trivial family if and only if $\dot{f}_s \in T\mathcal{K}_V \cdot f_s$, smoothly in $s$

**Theorem 3.2.2.**
Let $f \in \mathcal{E}^p$ be a germ and $V \subseteq \mathbb{R}^p$ a Variety. If

$$\mathfrak{m}^{k+1}\theta(f) \subseteq \mathfrak{m}_n T\mathcal{K}_V \cdot f$$

then $f$ is $k$-determined with respect to $\mathcal{K}_V$-equivalence.

The definitions of induced and versal deformations also follow from the $\mathcal{K}$-equivalence case in an obvious way, but in this setting infinitesimal versatility only implies versatility, with the converse not being true in general.

**Theorem 3.2.3.**
Let $V \subseteq \mathbb{R}^p$ be an algebraic Variety, $f \in \mathcal{E}^p$ a germ and $F : (\mathbb{R} \times \mathbb{R}^a, (0,0)) \to \mathbb{R}^p$ a deformation of $f$. If

$$T_e \mathcal{K}_V \cdot f + \dot{F} = \theta(f)$$

then $F$ is $\mathcal{K}_V$-versal.

One could go into more detail on all of this, but we sadly can't.

## §4. Final Words

Armed with all of these tools one is now finally able to define two bifurcations $G, F$ to be **path-equivalent** if $g_0 \sim_{\mathcal{K}} f_0$ and $h_G \sim_{\mathcal{K}_{\Delta_H}} h_F$ with $H$ being the $\mathcal{K}$-versal unfolding of $g_0$ and $f_0$ and $h_G, h_F$ such that $G \sim_{\mathcal{K}} h_g^* H$ and $F \sim_{\mathcal{K}} h_f^* H$, matching the intuition we developed at the beginning, that two bifurcations should be equivalent if their paths meet the discriminant in an equivalent way. Using commutative algebra it can even be shown that this notion of path-equivalence is equivalent to the more "naive" notion of $G, F$ being equivalent if $g_0 \sim_{\mathcal{K}} f_0$ and $G \sim_{\mathcal{K}_{\mathrm{un}}} F$ as unfoldings of $g_0, f_0$ (See [2] for a proof of this. Then one could classify bifurcations (up to a certain codimension at least) by first using the classification of their behavior at $\lambda = 0$ and then classifying the paths that induce them (For more on that see Chapter 21 of [1]). One could, but we sadly we can't.

## Bibliography

[1] J. Montaldi, *Singularities, bifurcations and catastrophes*. 2021.

[2] D. Mond and J. Montaldi, "Deformations of maps on complete intersections, Damon's Kv-equivalence and bifurcations," *Singularities*. Cambridge University Press, United States, pp. 263–284, 1994.