Holonomic approximation theorem, part 2

This is a slightly edited version of the notes I took for my seminar talk on the 16th; apologies for the still crude formatting. All mathematical content is taken from the chapters 3.7, 3.8, 4.1 and 4.2 of the second edition of Cieliebak-Eliashberg-Mishachev's book *Introduction to the h-principle.*

Recall that last week, our main result was the following theorem:
Holonomic approximation theorem:
Let $p : X \to V$ be a smooth fibre bundle, $A \subset V$ a polyhedron of positive codimension, and $F : \mathrm{Op}\, A \to X^{(r)}$ a smooth section of the r-jet bundle of p. Then you can find isotopies $h_t : V \to V, t \in I$ and holonomic sections $\tilde{F} : \mathrm{Op}\, h_1(A) \to X^{(r)}$ with h_t C^0-close to id and $\tilde{F}$ C^0-close to $F|_{\mathrm{Op}\, h_1(A)}$. Furthermore, if F is holonomic on some closed $B \subseteq A$, we can take $h_t = \mathrm{id}$ and $\tilde{F} = F$ near B.

We will derive some more general variants of that, then look at some simple concrete examples.

First, a parametric variant:
Parametric & relative holonomic approximation theorem:
Let $p : X \to V$ be a smooth fibration, $A \subset V$ a polyhedron of positive codimension, $B \subseteq A$ a subpolyhedron, and $F_z : \mathrm{Op}\ A \to X^{(r)}, z \in I^m$ a family of smooth sections such that F_z is holonomic for all $z \in \partial I^m$ and all F_z are holonomic on a neighbourhood of B. Then there exist families of diffeotopies $h_z^t : V \to V, t \in I, z \in I^m$ and holonomic sections $\tilde{F}_z : \mathrm{Op}\ h_z^1(A) \to X^{(r)}, z \in I^m$ such that $h_z^t(v) = v$ and $\tilde{F}_z(v) = F_z(v)$ for $(z, v) \in (I^m \times \mathrm{Op}\ B) \cup (\partial I^m \times A)$, and they can be chosen with h_z^t C^0-close to id_V and $\tilde{F}_z(v)$ C^0-close to $F_z(v)$ for all $z \in I^m, v \in \mathrm{Op}\ h_z^1(A)$.

Recall how we proved the holonomic approximation theorem last time. We inductively worked our way up the skeletons of our polyhedron A (first the points, then the edges etc.), allowing us to work at a single simplex at a time; then since that is contractible, we could find a local trivialisation and work in $J^i(\mathbb{R}^n, \mathbb{R}^q)$, with $n := \dim V$ and $q := \dim X - n$. The proof on $(\Delta^k, \partial\Delta^k) \simeq (I^k, \partial I^k)$ was then carried out using a lemma like this:

Holonomic approximation over a cube:
Let $k < n$, view I^k as $I^k \times \{0\}^{n-k} \subset \mathbb{R}^n$, and suppose $F : \mathrm{Op}\, I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ is a section that is holonomic on $\mathrm{Op}\, \partial I^k$. Then there exist a diffeomorphism

$$h : \mathbb{R}^n \to \mathbb{R}^n, (x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1}, x_n + \varphi(x_1, ..., x_n))$$

and a holonomic section $\tilde{F} : \mathrm{Op}\, h(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ such that $h = \mathrm{id}$ and $\tilde{F} = F$ on $\mathrm{Op}\, \partial I^k$, with h C^0-close to id and $\tilde{F}$ C^0-close to $F|_{\mathrm{Op}\, h(I^k)}$.

This is lemma 3.2.1 from the book; it is slightly more general than the lemma last week in that F doesn't need to vanish near ∂I^k. I won't prove it again nonetheless.

Analogously, for parametric holonomic approximation it also suffices to prove the result over a cube:

Parametric holonomic approximation over a cube:
Let $k < n$ and suppose $F_z : \operatorname{Op} I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q), z \in I^m$ is a smooth family of sections that are holonomic on $\operatorname{Op} \partial I^k$ for all $z \in I^m$ and holonomic on $\operatorname{Op} I^k$ for $z \in \operatorname{Op} \partial I^m$. Then there exists a family of diffeomorphisms

$$h_z : \mathbb{R}^n \to \mathbb{R}^n, (x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1}, x_n + \varphi_z(x_1, ..., x_n))$$

and holonomic sections $\tilde{F}_z : \operatorname{Op} h_z(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ such that:

- $h_z = \text{id}$ and $\tilde{F}_z = F_z$ on $\operatorname{Op} \partial I^k$ for all $z \in I^m$,
- $h_z = \text{id}$ and $\tilde{F}_z = F_z$ for all $z \in \partial I^m$,
- h_z is C^0-close to id,
- $\tilde{F}_z$ is C^0-close to $F_z|_{\operatorname{Op} h_z(I^k)}$.

Proof: Let $J^r(\mathbb{R}^{m+n}|\mathbb{R}^n, \mathbb{R}^q)$ denote the bundle $\mathbb{R}^m \times J^r(\mathbb{R}^n, \mathbb{R}^q) \to \mathbb{R}^m \times \mathbb{R}^n$. The family of sections $F_z : I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ can then viewed as a single section $\overline{F} : \operatorname{Op} I^{m+k} \to J^r(\mathbb{R}^{m+n}|\mathbb{R}^n, \mathbb{R}^q)$, which in turn lifts along $\pi : J^r(\mathbb{R}^{m+n}, \mathbb{R}^q) \to J^r(\mathbb{R}^{m+n}|\mathbb{R}^n, \mathbb{R}^q)$ to a section $\overline{\overline{F}} : I^{m+k} \to J^r(\mathbb{R}^{m+n}, \mathbb{R}^q)$ that can be chosen / extended to be holonomic near ∂I^{m+k}. Applying holonomic approximation over I^{m+k}, we can get an approximation $\tilde{\tilde{F}}$ of $\overline{\overline{F}}$ on a perturbed cube $h(I^{m+k})$ for a diffeomorphism

$$h : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}, (x_1, ..., x_{m+n}) \mapsto (x_1, ..., x_{m+n-1}, x_{m+n} + \varphi(x_1, ..., x_{m+n})).$$

Then $\tilde{F} := \pi \circ \tilde{\tilde{F}} : h(I^{m+k})$ can be viewed as the required family $\{\tilde{F}_z\}$ of approximations of F_z near $\{h_z(I^k)\}$. □

This concludes the proof of the parametrised holonomic approximation theorem. There is also a leafwise version for foliated manifolds; I skipped over it in the talk because I was short on time and didn't need it for applications, but will leave my notes here just in case.

First a quick rundown of a few prerequisites: let $\mathcal{F}$ be a foliation on V with leaves $\mathcal{L}_\alpha$. A submanifold $M \subseteq V$ is *transverse* to $\mathcal{F}$ if it is transverse to all of its leaves - or, if that isn't possible because $\dim M + \dim \mathcal{F} < \dim V$, if $T_x M \cap T_x \mathcal{L} = \{0\}$ for all $x \in M$. For each foliation, there is a *leafwise r-jet extension* $p^r_{\mathcal{F}} : X^{(r)}_{\mathcal{F}} \to V$ by considering r-tangency only along the leaves, and there is a canonical projection $p_{\mathcal{F}} : X^{(r)} \to X^{(r)}_{\mathcal{F}}$ factoring $p^r : X^{(r)} \to V$ as $p^r_{\mathcal{F}} \circ p_{\mathcal{F}}$. The *leafwise jet extension* $J^r_{f|\mathcal{F}}$ of sections $f : V \to X$ is defined as $p_{\mathcal{F}} \circ J^r_f$, and sections of this form are called *leafwise holonomic*.

Foliated holonomic approximation over a cube:
Let $k < n$, view $\mathbb{R}^k$ as $\mathbb{R}^k \times \{0\}^{n-k} \subseteq \mathbb{R}^n$ and suppose that $\mathbb{R}^n$ is equipped with a foliation $\mathcal{F}$ transversal to $\mathbb{R}^k$. Then for any section $F : \operatorname{Op} I^k \to X^{(r)}_{\mathcal{F}}$ that is leafwise holonomic near ∂I^k, there exist a leafwise diffeotopy $h^t : \mathbb{R}^n \to \mathbb{R}^n, t \in I$ and a leafwise holonomic section $\tilde{F} : \operatorname{Op} h^1(I^k) \to X^{(r)}_{\mathcal{F}}$ such that $h^t = \text{id}$ and $\tilde{F} = F$ on $\operatorname{Op} \partial I^k$, with h^t C^0-close to id and $\tilde{F}$ C^0-close to F.
Proof: In the case $k \leq \operatorname{codim} \mathcal{F}$ this is simply the fact that sections admit holonomic extensions pointwise (take $h^t := \text{id}$ and extend $F|_{I^k}$ leafwise holonomically

to OpI^k). Otherwise, lift F to a section $G : \operatorname{Op} I^k \to X^{(r)}$, change coordinates such that the foliation contains everywhere coordinate direction x_n, use holonomic approximation over a cube to get a diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ and a holonomic section $\tilde{G} : \operatorname{Op} h(I^k) \to X^{(r)}$, and take $h^t := (1-t)\mathrm{id} + th$ as the leafwise diffeotopy and $p_{\mathcal{F}} \circ \tilde{G}$ as the leafwise holonomic section $\tilde{F}$. □

This again also implies a version for arbitrary fibrations $p : X \to V$.
Foliated holonomic approximation theorem:
Let $p : X \to V$ a smooth fibre bundle whose base V is equipped with a foliation $\mathcal{F}$, and let $A \subset V$ be a polyhedron of positive codimension such that all of its strata (i.e. the interiors of the faces, the strata of the canonical stratification on any polyhedron) are transversal to $\mathcal{F}$. Furthermore, let $B \subseteq A$ be a closed subset and $F : \operatorname{Op} A \to X^{(r)}_{\mathcal{F}}$ a section that is leafwise holonomic near B. Then there exist a leafwise diffeotopy $h^t : V \to V, t \in I$ and a leafwise holonomic section $\tilde{F} : \operatorname{Op} h^1(A) \to X^{(r)}_{\mathcal{F}}$ such that $h^t = \mathrm{id}$ and $\tilde{F} = F$ near B, with h^t C^0-close to id and $\tilde{F}$ C^0-close to F.

Now we finally get to two examples showcasing the utility of the parametric holonomic approximation theorem.

Example (regular maps on an annulus):
Let $V := B_{r,R} = \{(x,y) \in \mathbb{R}^2 \mid r < \sqrt{x^2+y^2} < R\}$ be an annulus, $r < 1 < R$. Then there exists a smooth family $f_t : V \to \mathbb{R}, t \in I$ with $f_0(x,y) = x^2+y^2$, $f_1(x,y) = -x^2-y^2$ such that no f_t has a critical point.
[Sketch: two rounded cones / bowls with the center missing, the second one upside-down.]
Proof: The 1-jet space $J^1(V,\mathbb{R})$ is $V \times \mathbb{R} \times \mathbb{R}^2$, and holonomic sections are those of the form $J^1_f = (\mathrm{id}_V, f, \operatorname{grad} f)$. Identify $\mathbb{R}^2$ with $\mathbb{C}$; then since $\operatorname{grad} f_0 = -\operatorname{grad} f_1$,

$$F_t := (\mathrm{id}_V, (1-t)f_0 + tf_1, e^{i\pi t} \operatorname{grad} f_0), t \in I$$

is a smooth homotopy $J^1_{f_0} \rightsquigarrow J^1_{f_1}$, holonomic for $t = 0, 1$. Applying parametric holonomic approximation with $A := S^1$, we get a family of holonomic sections $\tilde{F}_t = J^1_{\tilde{f}_t} : U_t \to J^1(V,\mathbb{R})$ for U_t neighbourhoods of perturbed circles, with $U_t = V$ and $\tilde{F}_t = F_t$ for $t = 0, 1$. Furthermore, for $\tilde{F}_t$ close enough to F_t, $\tilde{f}_t$ has no critical points because $\operatorname{grad} \tilde{f}_t \approx e^{i\pi t} \operatorname{grad} f_0 \neq 0$. Let $\varphi_t : V \to V$ be an isotopy with $\varphi_0 = \varphi_1 = \mathrm{id}_V$ and $\varphi_t(V) \subseteq U_t$; then $g_t := \tilde{f}_t \circ \varphi_t$ is the wanted family of functions.□

It is not too hard to find an explicit homotopy either; I won't spoil it here. One interesting observation though: you can't do it without breaking most of the symmetry; if your homotopy is e.g. mirror-symmetric, it won't work. That seems to be a common theme at least for these simple examples.

Example (sphere eversion):
Let $V := \{x \in \mathbb{R}^3 \mid 1-\varepsilon < |x| < 1+\varepsilon\}$ denote an ε-thickened sphere in $\mathbb{R}^3$, $i_V : V \hookrightarrow \mathbb{R}^3$ the inclusion and inv the inversion map

$$\mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}, x \mapsto \frac{(x_1, x_2, -x_3)}{|x|^2}.$$

Then $\mathrm{inv} \circ i_V : V \to \mathbb{R}^3$ is regularly homotopic (i.e. homotopic through immersions) to i_V.
Note: the $-x_3$ is necessary, because otherwise inv would be orientation-reversing.
Proof: Let $f_0 := \mathrm{inv} \circ i_V$, $f_1 := i_V$ and note $J^1(V, \mathbb{R}^3) \simeq V \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$, $J^1_{f_i} = (\mathrm{id}_V, f_i, df_i)$. df_0 and df_1 both take value in $\mathrm{GL}_+(3)$, so $J^1_{f_0}$ and $J^1_{f_1}$ are homotopic because $\pi_2(\mathrm{GL}_+(3)) \simeq \pi_2(\mathrm{SO}(3)) \simeq 0$; let F_t be a smooth homotopy (also easy to construct directly). Applying parametric holonomic approximation with $A := S^2$, we get a family of holonomic sections $\tilde{F}_t = J^1_{\tilde{f}_t} : U_t \to J^1(V, \mathbb{R}^3)$ for U_t neighbourhoods of perturbed spheres, with $U_t = V$ and $\tilde{F}_t = F_t$ for $t = 0, 1$. Like before, for $\tilde{F}_t$ close enough to F_t we have $d\tilde{f}_t$ invertible so $\tilde{f}_t$ is a regular homotopy, and we can compose it with an isotopy $\varphi_t : V \to V$ with $\varphi_t(V) \subseteq U_t$ and $\varphi_0 = \varphi_1 = \mathrm{id}_V$. □

In fact, something stronger is true: by the same argument, every orientation-preserving immersion $V \to \mathbb{R}^3$ is regularly homotopic to i_V. Since every immersion $S^2 \to \mathbb{R}^3$ can be extended to an orientation-preserving immersion $V \to \mathbb{R}^3$, all immersions $S^2 \to \mathbb{R}^3$ are regularly homotopic.

Also, this is again not possible without breaking most of the symmetry: with a mirror symmetry you would have to turn a circle inside-out, which is impossible, and with a continuous rotational symmetry around an axis it also doesn't work out. Discrete rotational symmetries are possible though, like the famous visualisations of the result show.