The h-principle and $\mathcal{R}$-holonomic approximation

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Abstract

In this talk we first review what has been done so far in the seminar: the holonomic approximation theorem, h-principles for open diff-invariant differential relations (e.g. Smale-Hirsch) and h-principles for symplectic and contact structures on open manifolds. Before moving on to the main part of the talk, we mention what the rest of the seminar will do. The main dish is an extension of the holonomic approximation theorem that does not require the differential relation to be open. We explain the necessary conditions and how to adapt the proof of the usual theorem to this one before finishing with some applications.

1 Context

Consider a smooth fiber bundle $X \longrightarrow V$ and recall that $\mathcal{R}$ usually denotes a differential relation, i.e. $\mathcal{R}$ is a subset of the r-th jet-space $X^{(r)}$ of (sections of) X. Recall that a section of $X^{(r)} \to V$ is called holonomic if it's the r-jet of a section of $X \to V$. Given a differential relation $\mathcal{R}$, a *formal solution* is a section of $X^{(r)} \longrightarrow V$ valued in $\mathcal{R}$ and a *genuine solution* is such a section that is holonomic.

Interpretation of this: Think of immersions, the space of formal solutions is just the linear monomorphisms of the tangent bundles, which may or may not be the differential of a function, i.e. a genuine solution. Understanding these injective maps is linear algebra on smooth manifolds, which is quite algebro-topological in nature. However, understanding the space of actual solutions is very complicated a priori.

***H*-princpiple:** A differential relation $\mathcal{R}$ is said to satisfy the h-principle if the natural inclusion of solutions into formal solutions is a (weak) homotopy equivalence.

- In particular, this means that given a formal solution, we can connect it (i.e. homotope it) in $\mathcal{R}$ (i.e. through formal solutions) to a genuine solution. We also have that paths of formal solutions in $\mathcal{R}$ that join to genuine solutions can be deformed inside $\mathcal{R}$ to a path of genuine solutions keeping the endpoints fixed. This corresponds to the surjectivity and injectivity of the inclusion at the level of π_0 respectively. We could keep going with the higher homotopy groups. This is what the book calls the parametric h-principle, usually just regarded as the h-principle.

- We also have "local" versions of the (parametric) h-principle: satisfying the h-principle near some $A \subseteq V$ will mean having formal and genuine solutions defined on some open neighbourhood of A as well as all of the homotopies to happen withtin the open set.

- Recall as well that the relative h-principle is as above but requiring the homotopies involved to stay fixed in a fixed subset where the formal solutions are genuine.

- The book also defines the $\mathcal{C}^0$-dense h-principle: it requires formal solutions to be connected in $\mathcal{R}$ to genuine ones in such a way that the base of this path (i.e. the underlying sections of $X \to V$) stays within an arbitrarily small neighborhood of the base of the original formal solution.

Goal of the seminar: We want to describe natural and general conditions on $\mathcal{R}$ that ensure it satisfies the h-principle and then use this to study immersion theory for open and closed manifolds, as well a similar theory in the symplectic and contact setting. There are two main ways to do the first thing: holonomic approximation and convex integration.

So far, we have proven all forms of the local h-principle near a positive codimensional subset for Diff-inviariant open differential relations through the holonomic approximation theorem. We shall ellaborate on this next section but before, here are the consequences we have extracted from these:

1. *Open Diff-invariariant differential relations on open manifolds satisfy the h-principle.* The trick here was to observe that open manifolds can be compressed into their cores, which are positive codimensional and the above local h-principle works. The compression, however, breaks the $\mathcal{C}^0$-denseness property.

2. *Immersions and submersions of open manifolds: the h-principle holds for these.* This follows from the fact that the differential relations defined by immersions and submersions are open and diff-invariant.

3. *Embeddings of open manifolds directed by open sets satisfy the h-principle.* The proof is a clever (one gets embeddings) application of the holonomic approximation theorem to approximately integrate a deformation of the Gauss map of an embedding.

4. *Smale-Hirsch theorem: The h-principle for positive-codimensional immersions of closed manifolds holds.* This refers to the parametric version but the relative version also holds and even with the $\mathcal{C}^0$-density property. This follows from the previous theorem via the **micro-extension trick**: h-principle for immersions of positive codimension is the h-principle for their tubular neighbourhoods, which are open manifolds. *Smale's sphere eversion is a corollary of this.*

5. *Immersions transverse to distributions for closed manifolds: as long as the dimension of the domain plus the dimension of the target distribution are strictly smaller than the dimension of the target, the h-principle holds for immersions transverse to a distribution.* This is similar to Smale-Hirsch. One can also show the analogous result for sections of a fibration that are transverse to a distribution on the total space (under the same dimension condition). This requires a slight modification of the diff-invariance open theorem.

6. *H-principle for symplectic and contact structures on* ***open*** *manifolds.* This required understanding certain approximation theorems of differential forms handled via the holonomic approximation theorem. One should note that the analogue for **closed** manifolds is a lot, lot, lot harder: for closed contact manifolds there is an h-principle for *over-twisted* contact structures, leaving an interesting class of so called *tight* contact structures that do not obey the h-principle; for closed symplectic manifolds very little is known aside from the fact that the h-principle fails strongly. Philosophically, this is very interesting and I would say is one of the main mysteries of symplectic topology. In current years, the extend of flexibility for open manifolds has been worked on a lot, being able to handle symplectic invariants of so called Lioucille-sectors through (hard) homotopy theory. Conjecturally, all stable infinity categories are the Fukaya category of an (open) symplectic manifold, which showcases much flexibility and promises applications in motivit and cromatic homotopy theory.

7. *Isosymplectic and isocontact embeddings.* We have not really studied this, but given the tools at our current disposal we may as well have. The idea is that a type of h-principle (the surjectivity of π_0) holds for isosympl/contc embeddings of comdimension at least 2 in the open case and 4 in the closed case. By a micro-extension type trick (isotropic implies cotangent extension is isosympl/contact) one can show the h-principle for *subcritical* isotropic embeddings. Here the subcriticality is crucial, as for the critical ones (Lagrangians and

Legendrians) *embeddings* do not behave flexibly in general. The proof of these requires work to combine the h-principle for directed embeddings in the context of symplectic and contact maps with stability results of symplectic and contact structures: deform the maps through other maps that induce symplectic structures on the source in a careful enough way that Moser or Gray tell you that these structures remain essentially unchanged.

Let us sketch what we want to do **from now on**.

- Prove a holonomic approximation theorem (refered to as $\mathcal{R}$-holonomic approximation) that allows us to prove the local h-principle near positive codimensional subsets for Diff-invariant relations that are not necessarily open. For example, the $\mathcal{R}$ defined by isosymplectic, isocontact and symplectic or contact isotropic immersions are not open. This we will do today.

- A direct consequence of this new approximation theorem we can deduce: *the h-principle for isotropic immersions of open symplectic (with a cohomological condition) or contact manifolds and for subcritical immersions of closed manifolds* via a micro-extension trick (this is a weaker form of the harder to prove results stated before); *the h-principle for maps transverse to a contact distribution* (with no assumptions on dimensions). This we will also do today.

  Remark It is good to stop and ponder why these theorems we are using say something about immersions but not embeddings, while at some point in the book an h-principle if proven for embeddings.

- Further generalize the $\mathcal{R}$-holonomic approximation theorem to allow for invariance under a smaller but capacious enough subgroup of the diffeos of V. This will allow for a more h-principly proof of the results on isotropic immersions positive codimensional isosympl/contact immersions (note: not embeddings now), this includes the *h-principle for Lagrangian and Legendrian immersions of closed manifolds*. This will happen next talk.

  Remark As already mentioned in passing, the fact that Lagrangian embeddings behave much more rigidly than immersions is a crucial fact in symplectic topology, where the embedded Lagrangians encode deep information (e.g. a "silly" example is that that there are no embedded Lagrangian Klein bottles in $\mathbb{C}^2$ proves the square peg proglem). The theory of obstructing the existence of Lagrangians (even if formal solutions exist) is rich and interesting. In fact, it could be said that the failure of the h-principle here implies the existence of symplectically exotic $\mathbb{R}^{2n}$.[1]

- The seminar will end with the other known way to prove h-principles: convex integration. So-called ample differential relations will satisfy the h-principle and this shall be employed for further studying immersion theory (e.g. Nash-Kuiper).

- The last talk of the seminar will probabably be a survey on how Stein and Weinstein manifolds (afine complex manifolds and the symplectic counterpart) behave flexibly (exploiting much of the work done so far).

Remark. I was a bit sad writing all of this. Presented like this, it is easy to ignore all of the individual history each of these problems has and consequently underappreciate the power of these statements which are moreover treated with a "unified" method.

[1] More concretely: sometimes one can "resolve" an exact closed immersed Lagrangian in $(\mathbb{R}^{2n}, \omega_{\text{std}})$ (using h-principle ideas), producing a symplectic form ω on $\mathbb{R}^{2n}$ with an exact embedded Lagrangian, which can only mean that ω and ω_{std} are not equivalent. This is because there are no closed exact Lagrangians embedded in the standard $\mathbb{R}^{2n}$ by another hallmark result of Gromov using the theory of holomorphic curves.

2 Holonomic approximation theorem

We recall the motivation and ideas behind the holonomic approximation theorem we are already familiar with. This theorem was introduced by Eliashberg and Mishachev to formalize or exaplain some aspects of the proof of Gromov's h-principle for microflexible relations.

Given a section of the jet bundle, can we find a nearby holonomic section?

Example 1. The answer is no, for example consider $f(x) = x$ as a function $\mathbb{R} \longrightarrow \mathbb{R}$ and the section $(x, f(x), 0)$ of its 1-jet bundle, we cannot find a function g such that $(x, g(x), g'(x))$ is $\mathcal{C}^0$-close to the above. We can only accomplish this at a point. DRAW THIS.

That being said, slightly generalizing this example, we get close to the correct question to ask.

Example 2. Set $f(x, y) = x$ and consider the section $(x, y, f(x, y), 0, 0)$ (draw: graph of f in $\mathbb{R}^3$ along with the plane spanned by $(1, 0, 1)$ and $(1, 0, 0)$ for the 1-jet information). Again, it is not possible to find a holonomic approximation but if we consider the subset over $(x, y = 0)$ and $0 \leq x \leq 1$, we see some hope: we want to go up a hill by walking aaalmooost horizontally. Goat know how to do this well. Indeed, we make a lot of small wiggles to this interval. DRAW THIS.

So, it seems as though that we can find holonomic approximations of sections near positive-codimensional submanifolds if we are allowed to make a $\mathcal{C}^0$-perturbation.[2] This is the usual statement of the holonomic approximation theorem. Moreover, we can do this procedure in the relative sense. We state the version over the cube, which implies the full version doing induction over the skeleton and using that the fibration is trivial over simplices.

Theorem 3. *Let $k < n$ and $I^k \subseteq \mathbb{R}^k$ denote the unit cube considered in $\mathbb{R}^k \times 0^{n-k} \subseteq \mathbb{R}^n$ and suppose given a section*

$$F : OpI^k \longrightarrow \mathcal{J}^r(\mathbb{R}^n, \mathbb{R}^q)$$

that is already holonomic in $Op\partial I^k$. Then we can find arbitrarily small $\epsilon, \delta > 0$, a δ $\mathcal{C}'$-small diffeomorphism

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ \ (x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_{n-1}, x_n + \varphi(x_1, \ldots, x_n)),$$

a neighbourhood $Oph(I^k) \subseteq OpI^k$ and a holonomic section

$$\tilde{F} : Oph(I^k) \longrightarrow \mathcal{J}^r(\mathbb{R}^n, \mathbb{R}^q)$$

such that $h = id$ and $\tilde{F} = F$ in $Op(\partial I^k)$; and that $\tilde{F}$ is ϵ C^0-close to F.

Remark 4. It is important to note that this perturbation we construct in the theorem will be $\mathcal{C}^0$-small but $\mathcal{C}^1$-large.

Before sketching the proof, let us understand how it helps with the h-principle. Say that we have some differential relation $\mathcal{R}$ on the r-th jet space of $X \to V$ and a section $F : Op(A) \to \mathcal{R}$, and we want to find a homotopy of such sections to a holonomic one $G : Op(A) \to \mathcal{R}$ (showing a basic h-principle). We can certainly find a very small diffeotopy h^τ of V such that near $A' = h^1(A)$ we can find a holonomic section $F' : OpA' \to \mathcal{R}$ very close to $F|_{OpA'}$ (F' will indeed map into $\mathcal{R}$ if it is close to F and $\mathcal{R}$ is open for example)). Doing this for small enough parameters (this works if $\mathcal{R}$ is open for example) we can assume that the linear homotopy F'_t interpolating F and F' in OpA' is contained in $\mathcal{R}$. It could look like we are done, but we are definitely not: the holonomic section and homotopy need to be in OpA and not OpA'. We need a way to "straighten" the sections over A': A' is straightened by h^τ, so the sections will be straightned if we can manage to make h^τ on the jet-space (this is naturality) in a way that $\mathcal{R}$ is preserved (diff-invariance). Once we have this, the straightening of F' would be G and satisfy the desired properties. This shows (and the several other variations of holonomic approximations):

[2] Formally: Let $A \subseteq V$ be a polyhedron of positive codimension and F a section of $X^{(r)}$ on OpA. Then for arbitrariyly small $\delta, \epsilon > 0$ de can find a δ-small diffeotopy $(h^\tau)_{0 \leq \tau \leq 1}$ of V, a neighbourhood $Op\tilde{A} \subseteq OpA$ of $\tilde{A} := h^1(A)$ and a section of $X^{(r)}$ on $Op\tilde{A}$ that is $\mathcal{C}^0$-close to $F|_{Op\tilde{A}}$.

Corollary 5. *For a natural fibration $X \to V$, Diff-invariant and open differential relations satisfy the local h-principle near a polyhedron of positive dimension.*

- We say that a fibration $X \to V$ is **natural** if there is a section $j : \mathrm{Diff}(V) \to \mathrm{Diff}_V(X)$ of the natural projection $\mathrm{Diff}_V(X) \to \mathrm{Diff}(V)$, which gives the desired way $\mathrm{Diff}(V)$ can act on the jet spaces of X.

- For a natural fibration as above and an r-differential relation $\mathcal{R}$, we say that the relation is **Diff-invariant** if the action of $\mathrm{Diff}(V)$ preserves $\mathcal{R}$. One can show how this makes $\mathcal{R}$ into a subbundle of $X^{(r)}$ from which we can intuitively say that $\mathcal{R}$ can be defined in a coordinate-free way.

Now let us sketch the basic idea behing the holonomic approximation theorem.

Holonomic approximation theorem: sketch of proof. We want to formalize the wiggling procedure that allows animals to go up hills quite horizontally. When drawing the wiggle, we seem to be approximating finitely many lines perpendicular to the line we want to go up with, the more perpendiculars, the more wiggling. DRAW THIS. This perpendiculars are in fact holonomic approximations at points. The strategy emerges: we know how to holonomically approximate points, and we have a $I = [0,1]$-family of approximations, and by choosing finitely many and doing some kind of interpolation, we "glue" them to a holonomic approximation on I.

So, indeed, we want to approximate near a cube I^k in $\mathbb{R}^n$. We first approximate at each point of the cube, so we get an I^k-family of approximations. We then extend this to approximation over intervals I, getting a I^{k-1}-family of approximations, and so on. We just have to take care of doing this relative to the boundary (needed in the gluing, I think). In any case, we exemplify the proof in two key steps:

1. Construct holonomic approximations over points,
2. and the last step of the induction: given an I-family of holonomic approximations of F over the cubes $t \times I^{k-1}$ for $t \in I$, we "glue" them to a holonomic approximation of F over I^k.

The first thing is easy: recall that the jet-space at a point can be identified with polynomials after choosing some trivialization, and this suffices as we can do it locally. Here we are implicitly using that the differential relation is open.

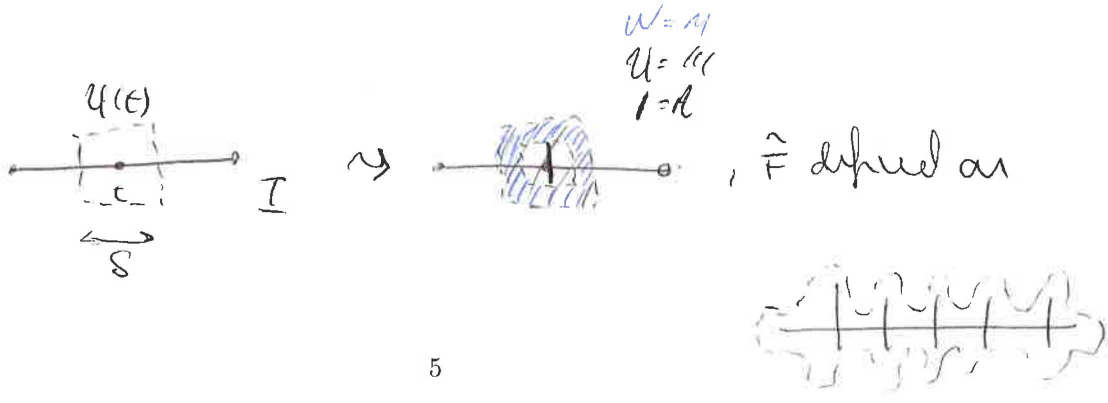
The second step is the hard part. The key property is that

$$\max_{t\in I,\, x\in U(t+\sigma)\cap U(y)} ||F_{t+\sigma}(x) - F_t(x)|| \xrightarrow{\sigma\to 0} 0,$$

which enables us to write down an **interpolation property**: *for any $\epsilon > 0$ there exists a $\sigma > 0$ and a family of holonomic sections $F_t^\tau : U(t) \times [0,\sigma]_\tau \to \mathcal{J}^r(\mathbb{R}^n, \mathbb{R}^q)$ for $t \in I$ such that for $\tau = 0$ we have F_t; for no τ or t F_t^τ and F_t are C^0-further than ϵ-apart; that for a given t, F_t^τ is fixed on $W(t)$ (drawing); and F_t^τ and F_t agree on $Op(t \times I^{k-1})$.*

With these we choose finitely many points $t \in I$ and and use this to construct the glued holonomic section on a modified neighbourhood, see the drawings.

n = 2
k = 1
l = 0



3 Holonomic $\mathcal{R}$-approximation theorem

We want to generalize this to differential relations that need not be open. The strategy is to find conditions that allow us to carry out the steps above. To extend over points we require that the Cauchy problem with initial data $(v, F(v))$ is locally solvable:

Definition 6. A differential relation $\mathcal{R}$ is locally integrable if for any section $F : v \to \mathcal{R}$ we can find a holonomic section $F' : Opv \to \mathcal{R}$ such that $F(v) = F'(v)$.

In fact, without further mention, we make use of a stronger version of this: a parametrized and relative version. The parametrized version is necessary to construct the homotopy for the local h-principle as well as for parametric and relative h-principles.

Example 7. Open differential relations are locally integrable. The isosympl/contact and isotropic differential relations are locally integrable. For example, if we have a section of $\mathcal{J}^1(V, W)$ at a single point, i.e. $(v, w, F : T_vV \to T_wW)$ such that F is injective and $F(T_vV)$ is a Lagrangian subspace, we can then use a Darboux chart on W near w: at the origin a Lagrangian $F(T_vV)$ has been selected, we can choose the same one in the nearby tangent spaces in a $\mathbb{R}^{\dim V}$-family of Lagrangians.

Example 8. As a non-example, consider Riemannian isometric immersions $V \to W$. This is due to curvature: even if we can identify two inner products at a point, curvature may not allow that extension.

To carry out the induction, the key step is the interpolation property that allow us to find the neighbourhood and the approximating section. The condition that will enable this is **micro-flexibility**. This will yield:

Theorem 9. *Let $k < n$ and $I^k \subseteq \mathbb{R}^k$ denote the unit cube considered in $\mathbb{R}^k \times 0^{n-k} \subseteq \mathbb{R}^n$ and suppose given a section*

$$F : OpI^k \longrightarrow \mathcal{R} \subseteq \mathcal{J}^r(\mathbb{R}^n, \mathbb{R}^q)$$

that is already holonomic in $Op\partial I^k$ and $\mathcal{R}$ is a locally integrable and micro-flexible differential relation. Then we can find arbitrarily small $\epsilon, \delta > 0$, a δ $\mathcal{C}'$-small diffeomorphism

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ \ (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_n)),$$

a neighbourhood $Oph(I^k) \subseteq OpI^k$ and a holonomic section

$$\tilde{F} : Oph(I^k) \longrightarrow \mathcal{R}$$

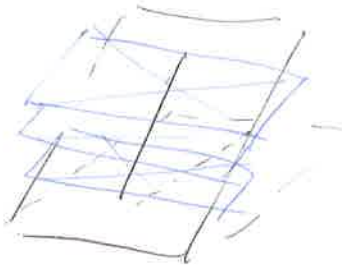
such that $h = id$ and $\tilde{F} = F$ in $Op(\partial I^k)$; and that $\tilde{F}$ is ϵ C^0-close to F.

As before, a direct consequence is the following local h-principle:

Corollary 10. *For a natural fibration $X \to V$,* Diff*-invariant, locally integrable and micro-flexible differential relations satisfy the local h-principle near a polyhedron of positive dimension.*

Proof. The proof is the same as before with one new complication: the linear homotopy between F and F' may not be contained in $\mathcal{R}$. The lazy approach to this is to assume that $\mathcal{R}$ is a local neighborhood retract, because then one can just compress the linear homotopy into $\mathcal{R}$ by the retraction. This is in fact sufficient for applications. The not so lazy applications consists of understanding the parametric local integrability and using it to construct the homotopy. $\square$

In the example before



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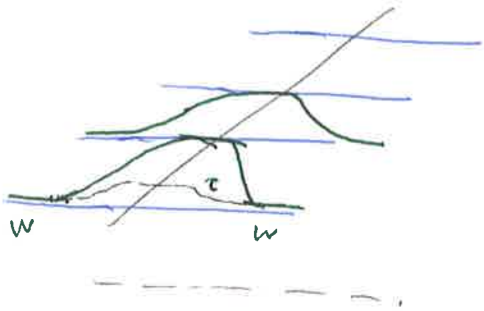

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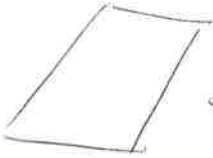

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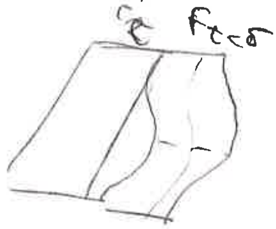
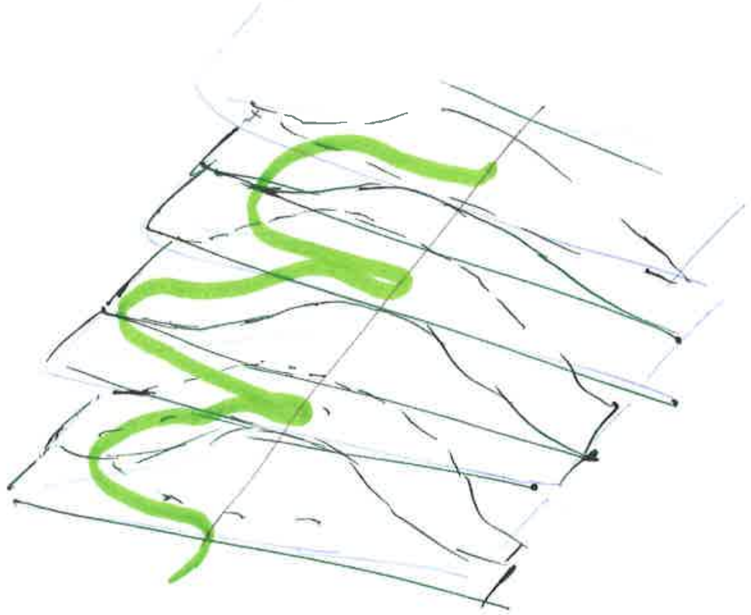
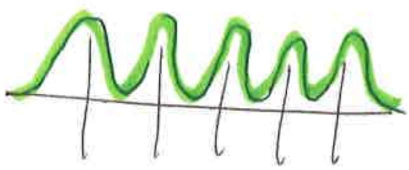




and this is the desired pleated behaviours pattern. This tells us that the polyhedron will be moved to



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will be what we look for.

3.1 Micro-flexibility

The restriction of a (Serre) fibration to an open set is not a fibration. The following definition is engineered to satisfy this property: it is a "short-time" homotopy lifting property.

Definition 11. A continuous map $X \to A$ is called a (Serre) micro-fibration if given the following diagram in Top

$$\begin{tikzcd}
0 \times B \arrow[r] \arrow[d, hook] & X \arrow[d] \\
I \times B \arrow[r, "f"] & A,
\end{tikzcd}$$

we can find an $0 < \epsilon \leq 1$ and a lift $\tilde{f}$ such that the following commutes

$$\begin{tikzcd}
& 0 \times B \arrow[r] \arrow[dl] \arrow[d, hook] & X \arrow[d] \\
{[0, \epsilon) \times B} \arrow[r, hook] \arrow[urr, dashed, "\tilde{f}"] & I \times B \arrow[r, "f"] & A.
\end{tikzcd}$$

Example 12. Fibrations are micro-fibrations. The inclusion of two open sets is a micro-fibration, and a homeomorphism if and only if it is a fibration. Restricting a micro-fibration to an open set is a micro-fibration. A non-proper submersion is a micro-fibration, and if proper a fibration.

A sheaf is called (micro-)flexible if the restriction maps on compact are Serre (micro-)fibrations. Hence, this defines the notion of (micro-)flexibility for partial differential relations asking the sheaf of holonomic sections to have these properties.

Example 13. If the restriction map betweent the neighbourhoods of two compact sets is a fibration and the h-principle is satisfied for those open sets, the relative h-principle is satisfied by the pair. This follows from the LES on homotopy for the fibrations and the five lemma (using the h-principle on four of the maps).

Remark 14. Gromov's way to think about the general theorem we are showing today was different than the Eliashberg-Mishacev method. The basic idea is that the h-principle in the presence of flexibility is "easy", in the sense that it is just topology. But then, he shows the following amazing theorem: *consider the topological ∞-site* Emb_n *of smooth n-manifold with open embeddings. Given an open manifold V, the following inclusion is an equivalence of ∞-categories*

$$\mathrm{Sh}^{\text{flexible}}(\mathrm{Emb}_n /V) \longrightarrow \mathrm{Sh}^{\text{micro-flexible}}(\mathrm{Emb}_n /V).$$

From this he can deduce the h-principle for Diff-invariant and micro-flexible relations.

To get the definition of micro-flexibility we have in the book we must consider two things: we consider pairs of compact sets diffeomorphic to $(K^n = [-1, 1]^n, K^k)$ to phrase the fibration property, and do it in a *relative* way, which I do not think is implied by the above definition. Their non-parametric flexibility is the path-lifting property for the restriction maps:

Definition 15. A differential relation $\mathcal{R} \subseteq X^{(r)}$ is called k-microflexible if for any sufficiently small open ball $U \subseteq V$, a pair $(I^n, I^k) \subseteq U$ (or a pair diffeo to it), a holonomic section $F : OpI^n \to \mathcal{R}$ and a holonomic homotopy $(F^\tau : OpI^k \to \mathcal{R})_{0 \leq \tau \leq 1}$ constant over $Op\partial B$ of the section $F^0 = F$ over OpB, we can find a number $0 < \epsilon \leq 1$ and a holonomic homotopy $(F^\tau : OpI^n \to \mathcal{R})_{0 \leq \tau \leq \epsilon}$ that is constant over $Op\partial A$ extending $(F^\tau : OpI^k \to \mathcal{R})_{0 \leq \tau \leq \epsilon}$. If ϵ can always 1 we call it k-flexible. When it holds for $0 \leq k \leq n-1$, we call it (micro-)flexible.

Here is a diagram description for the flexibility property (without capturing the condition of being constant near the boundary out of notational convenance):

$$\begin{array}{ccc} 0 & \xrightarrow[F]{} & \mathrm{Hol}(OpK^n, \mathcal{R}) \\ \big\downarrow & & \big\downarrow \\ [0,1] & \xrightarrow{(F^\tau)^\tau} & \mathrm{Hol}(OpK^k, \mathcal{R}) \end{array}$$

Example 16. The section sheaf of the jet-space is flexible and hence open differential relations are micro-flexible (this follows from the restriction property). These are not flexible.

Example 17. Symplectic and contact stability imply micro-flexibility of isocontact and isotropic immersions and k-micro-flexibility of isosymplectic and isotropic immersions for $k \neq 1$. These are not flexible.

Proof. Say we take the relation defining Lagrangian immersions. One must use Weinstein to look at Lagrangian embeddings near the zero section of the cotangent bundle of V rather than immersions into W. These sections can be thought of as one forms on V, and being Lagrangian is equivalent to being closed. Hence one has to check the question of micro-flexibility for the relation of a one-form being closed. As long as k is not 1, the Poincaré lemma provides the result: α a *closed* one form near K^n and α^τ a homotopy of *closed* 1-forms near K^k matching α near ∂K^k. We can find a family of functions f^τ near K^n such that $\alpha^\tau = df^\tau$ so $df^\tau - df^0 = 0$ near ∂K^k, which implies $f^\tau - f^0$ is locally constant near $\partial K^k \cong \mathbb{S}^{k-1}$, which will be connected (or empty) as long as $k \neq 1$, so locally constant implies constant. ☐

4 Immediate consequences

Using the local h-principle for micro-flexible differential relations along with the compression trick:

Theorem 18. *Let V be an open manifold and $X \to V$ a natural fiber bundle. Then any locally integrable and microflexible* Diff V*-invariant differential relation $\mathcal{R} \subseteq X^{(r)}$ satisfies the h-principle.*

Corollary 19. *The h-principle holds for isocontact and isotropic immersions on open contact manifolds. For closed manifolds, subcritical isotropic immersions also satisfy the h-principle.*

Proof. The first statement follows from the fact that we have verified that isocontact and isotropic immersions satisfy local integrability and micro-flexibility. The second follows from an application of the micro-extension trick. ☐

Corollary 20. *The h-principle holds for isosympletic and isotropic immersions on open symplectic manifolds. For closed manifolds, subcritical isotropic immersions also satisfy the h-principle.*

Proof. This is is similar as before with a couple of differences: in the definition of formal isotropic (isosymplectic) immersion one must add the cohomological condition tha the base map pulls back the symplectic form to an exact form (pulls back the cohomology classes of symplectic structures one to the other); and one must overcome the lack of micro-flexibility for $k = 1$. This is done by reducing the problem to a contact problem, next talk will explain this further. ☐

The holonomic approximation theorem was used to show that, under a dimensional condition, maps transverse to a distribution satisfy the h-principle. It turns out that for contact structures (and further, completely non-integrable distributions, though it is harder) the dimension condition can be dropped or much improved.

Theorem 21. *Let (M, ξ) be a (possibly closed) contact manifold. Then the maps $f : V \to M$ transverse to ξ satisfy the h-principle.*

Proof. Extend maps $f : V \to M$ transverse to ξ to maps $V \times \mathbb{R} \to M$ transverse to ξ but tangent to it along each fiber of $V \times \mathbb{R} \to V$. This defines a locally integrable and microflexible differential relation (the tangential part is locally integrable and microflexible while the transverse part is open). This differential relation is not $\mathrm{Diff}(V \times \mathbb{R})$-invariant but invariant under diffeos that project to the idenitty of V. These are the kinds we can prove a modified h-principle with as in last talk. $\square$

Theorem 22. *Let (M, ξ) be a (possibly closed) contact manifold and $\dim V < \dim M$. Then the immersions $f : V \to M$ transverse to ξ satisfy the h-principle.*

Proof. Same strategy as above but incorporating the immersion condition in the formal solutions of the extended problem (immersions of $V \times \mathbb{R}$ makes the dimension condition pop up). $\square$