Lagrangian and Legendrian immersions

The theorems on h-principle we have seen till now have required the PDR $\mathcal{R} \subset X^{(r)}$ to be Diff V-invariant. What can we say if $\mathcal{R}$ is invariant only wrt a subgroup $\mathfrak{U}$ of Diff V?

[$X \xrightarrow{\pi} V$ is the fibration, $\mathcal{R} \subset X^{(r)}$]

For example,

1) Let (V, ω_V), (W, ω_W) be sympletic manifolds. Then $\mathcal{R}_{isosymp} \subset J^1(V,W)$ is given by

$$\mathcal{R}_{isosymp} = \{(v, w, \Phi) \mid \Phi : T_vV \to T_wW \text{ injective}, \omega_W(\Phi(X), \Phi(Y)) = \omega_V(X,Y) \; \forall X, Y \in T_vV\} \subset J^1(V,W)$$

(PDR for isosymplectic immersions)

If $f:(V,\omega_V) \to (W, \omega_W)$ is an isosymplectic map & $\phi: V \to V$ is a diffeomorphism, then $(f \circ \phi)^* \omega_W = \phi^*(f^*\omega_W) = \phi^* \omega_V$ is equal to ω_V iff ϕ is a symplectomorphism.

$\therefore \mathcal{R}_{isosymp}$ cannot be Diff V-invariant. It is Symp(V)-invariant.
(Symplectomorphisms $V \to V$)

2) Similarly, given contact manifolds (V, ξ_V) & (W, ξ_W),

$$\mathcal{R}_{isocont} = \{(v, w, \Phi) \in J^1(V,W) \mid \Phi: T_vV \to T_wW \text{ injective}, \Phi^{-1}(\xi_W) = \xi_V, \Phi : \xi_V \to \xi_W \text{ induces a conformally sympl map}\}$$

(PDR for isocontact immersions)

is invariant wrt only contact diffeomorphisms of V.

To prove a holonomic approximation thm for such relations, (i.e. w/ $h^\tau \in \mathfrak{U}$ (see below)), it will be convenient to assume that the diffeomorphisms in $\mathfrak{U}$ are obtained via flows of compactly supported vector fields - this suggests taking $\mathfrak{U}$ to be a Lie subgroup of $\text{Diff}_c(V)$ (the Lie group of compactly supported diffeomorphisms of V)

$\phi: V \to V$ diffeo, $\phi|_{V \setminus K} = Id_{V \setminus K}$ for some K cpct.

These are infinite dimensional & their theory is subtle; we will only consider cases where $\mathfrak{U}$ is $Ham(V, \omega)$ (see below) or $\text{Diff}_{cont} V$, so we will take for granted that they are Lie subgroups of $\text{Diff}_c V$ with their Lie algebras as given below (see Rmk)

1) (V,ω) is a sympl. mfd

$Ham(V,\omega) = \{\phi \in \text{Diff}_c V \mid \phi = \phi_1$ st $\exists$ a family $\{\phi_t\}_{t \in [0,1]}$, $\phi_0 = id$, $\frac{d}{dt}\phi_t = X_t \circ \phi_t$ where X_t are hamiltonian vector fields wrt $H_t : V \to \mathbb{R}$, smooth maps. $\}$

(Hamiltonian diffeos of (V,ω)) comp. supp

$\mathfrak{ham} = \{X_H \mid H: V \xrightarrow{smooth} \mathbb{R} \text{ compactly supported}\}$

(Hamiltonian vector fields) comp. supp

[Recall: Given $H: V \xrightarrow{smooth} \mathbb{R}$, X_H is the unique vector field st $\omega(X_H, -) = -dH$]

2) $\mathrm{Diff}_{cont}(V,\xi) = \{\phi \in \mathrm{Diff}_c(V) \mid \phi$ is a contactomorphism$\}$ for a contact mfd (V,ξ)

$\mathrm{cont} = \{X \in \mathfrak{X}_c(V) \mid (\phi_X^t)^* \xi = \xi$ where ϕ_X^t is the flow of $X\}$

contact vector fields

This is equivalent to the condition that if $\xi = \ker\alpha$ in a nbhd, then $\mathcal{L}_X \alpha = f\alpha$ for some $f: V \to \mathbb{R}$

→ Suppose $\xi = \ker\alpha$ for some $\alpha \in \Omega^1(V)$ contact form
Let R be the Reeb vector field wrt α: the unique vf R st $\alpha(R)=1$ & $d\alpha(R,-)\equiv 0$

Then given $H: V \to \mathbb{R}$, $\exists! X_H \in \mathfrak{X}(V)$ st $\alpha(X_H) = -H$ & $\iota_{X_H} d\alpha = dH - (dH(R))\alpha$ and X_H is a contact vf. Moreover, every contact vector field is of this form.

To prove the hol. approx thm in this case, we will require that $\mathfrak{A}$ satisfies two additional properties (CAP1), (CAP2); such $(\mathfrak{A}, \mathfrak{a})$ are called capacious. (Lie algebra of vector fields)

Rmk. I think we don't really require the full strength of $(\mathrm{Ham}(V,\omega), \mathrm{ham})$ or $(\mathrm{Diff}_{cont}(V), \mathrm{cont})$ being Lie subgroups. What seems to matter is only that they satisfy the following 2 properties and that the flows of vector fields in these cases give diffeomorphisms in the respective subgroups of $\mathrm{Diff}_c(V)$

(CAP1) Given a vector field $v \in \mathfrak{a}$, a compact subset $A \subset V$ and a nbhd U of A in V, $\exists \tilde{v} \in \mathfrak{a}$ supported in U st $\tilde{v}|_A = v|_A$.

→ $(\mathrm{Ham}(V,\omega), \mathrm{ham})$ satisfies this: Given $X_H \in \mathrm{ham}$, $A \subset U \subset V$ as above, $\exists$ a cutoff function $u: V \to \mathbb{R}$ st $u|_{\tilde{U}} \equiv 1$ for some $A \subset \tilde{U} \subset U$ open & $u|_{V \setminus U} \equiv 0$.
Define $\tilde{H} = uH$.
Then $X_{\tilde{H}}$ is as required.

→ Similarly, $(\mathrm{Diff}_{cont}(V), \mathrm{cont})$ satisfies this.

(CAP2) Given $x_0 \in V$ and a tangent hyperplane $\tau \in T_{x_0}V$ at x_0, $\exists v \in \mathfrak{a}$ transverse to τ.

→ $(\mathrm{Ham}(V,\omega), \mathrm{ham})$ satisfies this: Given x_0, τ as above, choose a Darboux chart $(\mathcal{U}, x)$ near x_0. $\mathcal{U} \xrightarrow{x = (x^1, \dots, x^{2n})} \mathbb{R}^{2n}$. In $(\mathbb{R}^{2n}, \omega_{std})$, we have

$$\begin{cases} X_H = \partial_{x^{2i}} & \text{if } H = x^{2i-1}, \ i = 1,\dots,n \\ X_H = \partial_{x^{2i-1}} & \text{if } H = x^{2i}, \ i = 1,\dots,n \end{cases}$$

$x_* \tau$ is transverse to atleast one of these X_H's.
So we can define H so that it matches an appropriate coord fn in a nbhd (smaller) & has cpct support in $\mathcal{U}$.

→ Similarly, by picking a Darboux chart near a pt in a contact mfd & working in $(\mathbb{R}^{2n+1}, \xi_{std})$ we can prove this property for $(\mathrm{Diff}_{cont} V, \mathrm{cont})$

Local h-principle for microflexible $\mathfrak{A}$-invariant relations

Thm: Let $\mathfrak{A} \in \mathrm{Diff}_c V$ be a capacious subgroup, $X \to V$ be a natural fibration & $\mathcal{R}$ an $\mathfrak{A}$-invariant locally integrable, microflexible diff rel. ($\hat{X}^{(r)}$)
Then the local-h principle holds for $\mathcal{R}$ near any subpolyhedron $A \subset V$ of positive codim.
(other forms hold too if (CAP1), (CAP2) also holds parametrically)

→ In the previous talk, we obtained the local-h principle for locally integrable & microflexible diff rels via the holonomic R-approx thm. That method works here too, but since $\mathcal{R}$ is not Diff-V invariant, we will have to ensure that the diffeotopies h^τ are in $\mathfrak{A}$. Below is a sketch of the pf of the usual hol approx thm & how we can modify.

Sketch of pf of usual Hol approx thm:

I) Reduce to the case $(I^k, \partial I^k) = (A, B)$, $X^{(r)} = J^r(\mathbb{R}^n, \mathbb{R}^q)$

II) Consider I^k as a family of $\{y \times I^l\}_{y \in I^{k-l}}$ & show that if $\exists$ a holonomic ext extension along these cubes, we can glue to get hol extensions along larger cubes

a) $F : \mathcal{O}p\, I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ is hol over ∂I^k & holonomically extended along cubes $\{y \times I^l\}_{y \in I^{k-l}}$ $0 \le l < k$

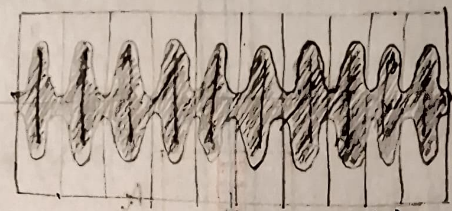
ie $\exists \delta > 0$ & $F_y = J^r_{f_y} : V_\delta(y) \to J^r(V_\delta(y), \mathbb{R}^q)$ $\forall y \in I^{k-l}$ st

- $F_y|_{y \times I^l \cup V_\delta(y)} = F|_{y \times I^l \cup V_\delta(y)}$
- $F_y = F|_{V_\delta(y)}$ for $y \in \mathcal{O}p\, \partial I^{k-l}$

Then $\forall \varepsilon > 0\ \exists N, \Omega_z, \tilde{F}_z : \Omega_z \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ for $z \in I^{k-l-1}$

st $\tilde{F}_z = F$ on $\Omega_z \cap \mathcal{O}p\, \partial I^k$ & $\| \tilde{F}_z - F|_{\Omega_z} \|_{C^0} < \varepsilon$

Ω_z is the shaded region here



$n=2, k=1, l=0$

$1/N$

Prove using Interpolation property

b) Induction step: Given U_l, a nbhd of $I^k \subset \mathbb{R}^n$, $F^l : U_l \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ holonomic over ∂I^k and holonomically extended along cubes $y \times I^l$, and $U(F^l)$ a nbhd of $F^l(U_l) \subset J^r(\mathbb{R}^n, \mathbb{R}^q)$, $\exists$ a diffeo

$h_{l+1} : \mathbb{R}^n \to \mathbb{R}^n$ obtained by wiggling I^k in the n-th coordinate, & supported in U_l

& $\tilde{F}^{l+1} : \tilde{U}_{l+1} \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ st

- $h_{l+1}|_{\mathcal{O}p\,\partial I^k} = Id$, $\tilde{F}^{l+1}|_{\mathcal{O}p\,\partial I^k} = F^l$
- $\tilde{F}^{l+1}(\tilde{U}_{l+1}) \subset U(F^l)$
- $\tilde{F}^{l+1}(\tilde{U}_{l+1}) \subset U(F^l)$
- $\tilde{F}^{l+1}$ is holonomically extendable along cubes $h_{l+1}(z \times I^{l+1})$

Prove by obtaining via (a) an h_{l+1} of that form supp in U_l st it is Id on $\mathcal{O}p\, \partial I^k$ & st for $z \in I^{k-l-1}$, $h_{l+1}(z \times I^{l+1}) \subset \Omega_z$.

(a) then gives an $\tilde{F}_z$ defined on $\mathcal{O}p\, h_{l+1}(z \times I^{l+1})$

Then get a diffeotopy via linear interpolation from Id.

III a) Show $\exists$ extensions over pts

b) Successively apply II for $l = 0, .., k-1$

Here, I) works as above

II) a) requires microflexibility for (relative) Interpolation Property

b) we should make sure $h_i^I \in \mathcal{H}$ and that $h_i'(I^k \cap \tilde{U}_i) \cap A = \phi$ $\forall i$ so that the image of $h_i|_{z \times I^i}$ is contained in Ω_z so that $\tilde{F}_z$ is defined there. See the next page.

III a) required local integrability

b) Works as before

II b)

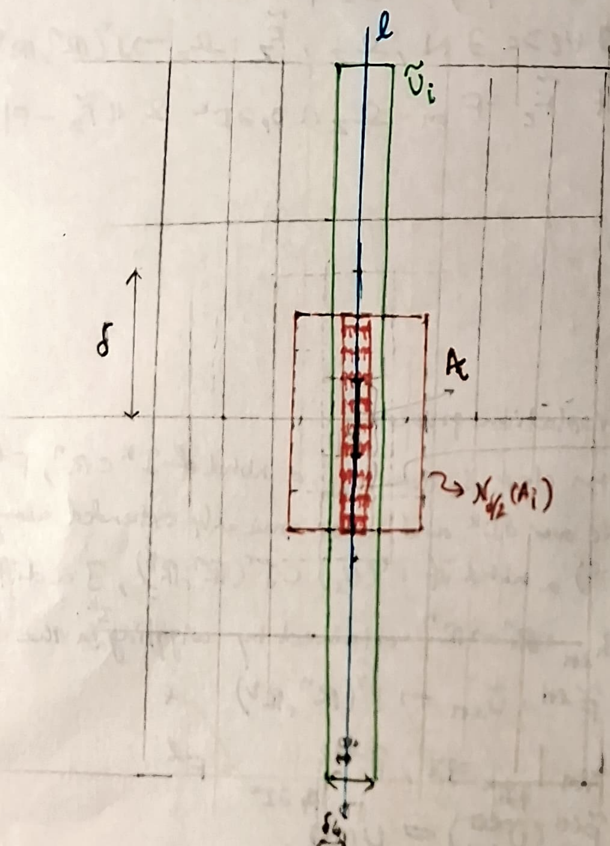
(CAP2) $\Rightarrow$ A can be subdivided st each of its simplices Δ admits a transverse vector field $V_\Delta \in \mathfrak{a}$

Near each simplex Δ, choose a coord system which identifies a slightly smaller domain U in the simplex w/ I^k & V_Δ w/ $\partial/\partial x_n$.

δ is st there is a family of holonomic sections

$F_y = J^r_{f_y} : U_\delta(y) \to J^r(U_\delta(y), \mathbb{R}^q)$

for $y \in I^{k-l}$



$n=2,\ k=1,\ l=0$

l is a line parallel to the x_n-axis which intersects A_i.

$\lambda = A_i \cap l$

$\tilde{\lambda} = \{-3\delta/4 \leq x_n \leq -\delta/4\} \cap l$

$A_i = U_{\delta/4}(c_i) \setminus V_\delta(c_i)$

$\tilde{U}_i = U_\delta(i\sigma) \cap ((i-1)\sigma, i\sigma) \times \mathbb{R}^{n-1})$

Using (CAP1) we can find $\tilde{v}_i \in \mathfrak{a}$ which coincides w/ v_i on

$$W := N_{\delta/2}(A_i) \cap \{c_i - \sigma/4 \leq t \leq c_i + \sigma/4\}$$

k is supported in a slightly larger subset of $\tilde{U}_i$. Let $\varphi_i^t := \exp(t\tilde{v}_i) \in \mathrm{Diff}(\tilde{U}_i)$

$\tilde{v}_i|_W = \frac{\partial}{\partial x_n}\Big|_W$.

So $\exp_p(\frac{\delta}{2}\tilde{v}_i) = p + \frac{\delta}{2} \underset{(0,0,\dots,0,1)}{\overset{\shortparallel}{e_n}}$

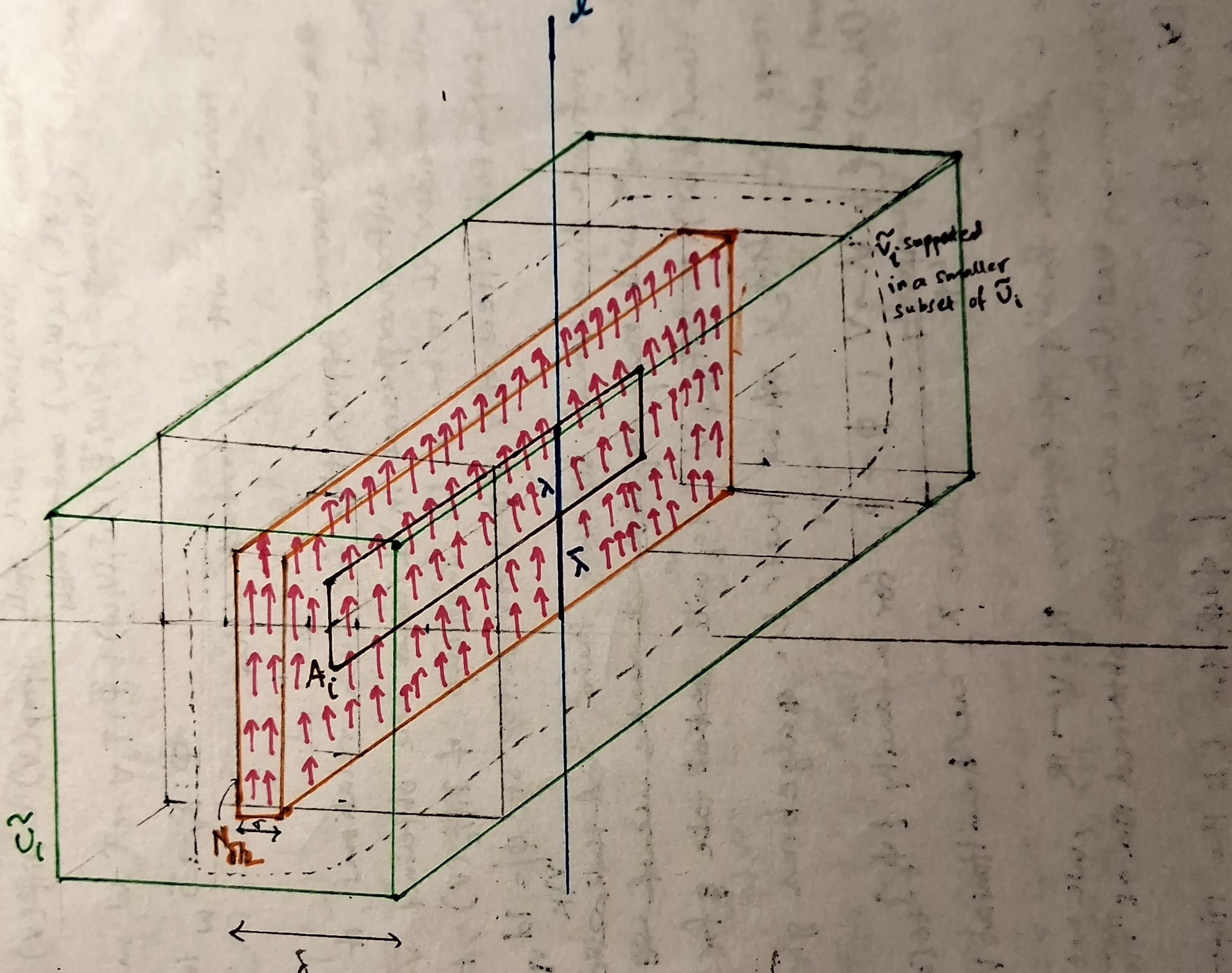
$\Rightarrow \varphi_i^{\delta/2}(\tilde{\lambda}) = \lambda$

$\therefore \varphi_i^{\delta/2}(I^k \cap \tilde{U}_i) \cap A_i = \emptyset$

Rescale so that this disjointness holds at time 1: $h_i^\tau := \varphi_i^{(\delta/2)\tau}$

$h_i^\tau \in \mathfrak{A}$ as $\tilde{v}_i \in \mathfrak{a}$.

The rest follows as before.



ℓ
$\tilde{v}_i$ supported in a smaller subset of $\tilde{v}_i$
λ
$\bar{\lambda}$
A_i
$\tilde{v}_i$
$N_{\delta/2}$
r
δ

Two useful lemmas

Lemma 1: Let (V, ω_V) be a symplectic mfd, $\pi: E \to V$ be a symplectic vector bundle over V. Then there is a symplectic structure $\tilde{\omega}$ on a nbhd $O_p V$ of the zero section $V \subset E$ st $\tilde{\omega}|_{TV} = \omega_V$ and the restriction of $\tilde{\omega}$ to the fibers E_v matches the symplectic structure ω_{E_v} from the symplectic VB.

Pf: Our idea will be to define a 2-form η which is closed, identically 0 on TV and matches ω_{E_v} on E_v and then take $\tilde{\omega}$ to be $\eta + \pi^* \omega_V$.

A sympl VB is locally of the form $\mathcal{U} \times \mathbb{R}^{2n}$ where the fibers have the standard sympl str ie $\forall v \in V$, $\exists \mathcal{U} \subset V$ an open set containing v and a fiberwise sympl $E|_{\mathcal{U}} \xrightarrow[\cong]{\Phi} \mathcal{U} \times \mathbb{R}^{2n}$. Cover V by a family of such open sets $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ w/ corr Φ_αs & define $\eta_\alpha := (pr_{\mathbb{R}^{2n}} \circ \Phi_\alpha)^* \omega_{std}$. By construction, the sympl str on fibers matches η_α & $\eta_\alpha|_{TV} = 0$. Let $\{\phi_\alpha\}_{\alpha \in I}$ be a partition of unity subordinate to $\{\mathcal{U}_\alpha\}_{\alpha \in I}$. Define $\eta := \sum_{\alpha \in I} \phi_\alpha \eta_\alpha$. Then $\eta|_{TV} = 0$ & $\eta|_{E_v} = \omega_{E_v}$ $\forall v \in V$.

But η need not be closed (even though η_α s are), which is a problem because we want $d\tilde{\omega} = 0 = d\eta$. So we construct $\tilde{\eta}$ from η which also satisfies the above properties.

Let $Z \in \mathfrak{X}(E)$ be the radial vector field on each fiber of π, ie on each fiber E_v, $v \in V$, for $y \in E_v$, identifying $T_y(E_v)$ w/ E_v, $Z(y) := y/2$.

The flow φ_Z^t of Z is fiberwise & satisfies $T\varphi_Z^t(X) = e^{t/2} X$ for $X \in E_v$ & $v \in V$

$\therefore (\mathcal{L}_Z \omega_{E_v})(X,Y) = \lim_{t \to 0} \frac{\omega(T\varphi_Z^t(X), T\varphi_Z^t(Y)) - \omega_v(X,Y)}{t} = \omega_{E_v}(X,Y) \lim_{t \to 0} \frac{e^t - 1}{t} = \omega_{E_v}(X,Y)$

$\therefore \mathcal{L}_Z \omega_v = \omega_v$. So $(\mathcal{L}_Z \eta)|_{E_v} = \mathcal{L}_Z \omega_{E_v} = \omega_{E_v} = \eta|_{E_v}$. $T\varphi_Z^t(X) = 0$ for $X \in T_v V$
$\Rightarrow \mathcal{L}_Z \eta|_{TV} = 0$ (1)

Expressing $d\eta$ in local coordinates, we get that $\iota_Z d\eta$ is 0 on fibers & on TV. (2)

Let $\tilde{\eta} := d\lambda$ where $\lambda = \iota_Z \eta$. Then $\tilde{\eta} = d(\iota_Z \eta) = \mathcal{L}_Z \eta - \iota_Z d\eta$

(1) & (2) $\Rightarrow \tilde{\eta}|_{E_v} = \omega_{E_v}$ & $\tilde{\eta}|_{TV} = 0$ and $\tilde{\eta}$ being exact, is closed.

We can now define $\tilde{\omega} := \tilde{\eta} + \pi^* \omega_V$, which will be closed as $\tilde{\eta}$ is,

$\tilde{\omega}|_{E_v} = \tilde{\eta}|_{E_v} + 0 = \omega_{E_v}$ and $\tilde{\omega}|_{TV} = \tilde{\eta}|_{TV} + \omega_V = \omega_V$.

The latter implies that $\tilde{\omega}$ is non-degenerate along $V \subset E \Rightarrow \tilde{\omega}$ is non-degenerate in a nbhd of V in E

$\therefore \tilde{\omega}$ is a sympl form on an open nbhd $O_p V$ of V in E, of the required form.

Lemma 2: Let (V, ξ_V) be a contact mfd & $E \xrightarrow{\pi} V$ be a symplectic vector bundle. Then there is a contact structure $\tilde{\xi}$ on a nbhd $Op\,V$ of V in E st $\tilde{\xi}_v = \xi_v \oplus E_v$ for $v \in V$ and the fibers with their symplectic structures are symplectic subspaces of $\tilde{\xi}_V$ wrt $CS(\tilde{\xi}_V)$.

Pf: We consider the case when $\xi_V = \ker \alpha$ for some $\alpha \in \Omega^1(V)$. The idea will be similar to that of Lemma 1. In the proof of Lemma 1, we constructed a closed (exact too) $\tilde{\eta} \in \Omega^2(E)$ satisfying $\tilde{\eta}|_{TV} = 0$ and $\tilde{\eta}|_{E_v} = \omega_{E_v}$. In fact, $\tilde{\eta}$ also satisfies $\tilde{\eta}(X, Y) = 0$ $\forall X \in T_v V$ $\forall Y \in T_v E$ $\forall v \in V$

Let $H : [0,1] \times E \to E$ be the htpy $H(t, e) = te$ from $\underset{H_1}{Id : E \to E}$ to $\underset{H_0}{\pi : E \to E}$

Then $\underset{[\tilde{\eta}]}{[H_1^* \tilde{\eta}]} = [H_0^* \tilde{\eta}] = [\pi^* \tilde{\eta}] = \underset{\text{as } \tilde{\eta}|_{TV} = 0}{0} \in H^2_{dR}(E)$ $\rightarrow \tilde{\eta}$ is exact.

We can also ensure that $\tilde{\eta}$ is exact with $\tilde{\eta} = d\beta$ for some β which vanishes on $TE|_V$

[Recall the prism operator $P : \Omega^2(E) \to \Omega^1(E)$ (here) used in the pf of htpy-invariance property of H^*_{dR} which satisfies $P(d\tilde{\eta}) + d\underbrace{(P\tilde{\eta})}_{-\beta} = H_0^* \tilde{\eta} - H_1^* \tilde{\eta} = -\tilde{\eta}$

Let $X \in T_v E$.

$$P\tilde{\eta}(X) := \int_0^1 (H^* \tilde{\eta})_{(t,v)} (\partial_t, X)\, dt = \int_0^1 \tilde{\eta}\big(\underbrace{T_{(t,v)} H(\partial_t)}_{\overset{\shortparallel}{0} \text{ as } v \in V}, T_{(t,v)} H(X)\big) dt = 0$$]

Define $\tilde{\alpha} := \beta + \pi^* \alpha$ (Say $\dim V = 2n+1$, $\dim E_v = 2q$ $\forall v \in V$)

$d\tilde{\alpha} = \tilde{\eta} + \pi^* d\alpha$

On V: $\tilde{\alpha} \wedge (d\tilde{\alpha})^{n+q} = \pi^* \alpha \wedge (\tilde{\eta} + \pi^* d\alpha)^{n+q} = \underbrace{\pi^* \alpha \wedge (\pi^* d\alpha)^n}_{\substack{\text{On } TV, \text{ it gives a} \\ \text{volume form by defn} \\ \text{\& is zero on fibers}}} \wedge \underbrace{\tilde{\eta}^q}_{\substack{\text{volume} \\ \text{form on fibers as } \tilde{\eta} \\ \text{is a sympl form there}}}$ (this is the only term that survives)

$\neq 0$

$\therefore \tilde{\alpha}$ gives a contact form on a nbhd of $V \subset E$.

Choosing a connection on E, we obtain an isomorphism $TE \cong HE \oplus VE$ using which we can write $Y \in TE$ as (X, F) for $X \in HE$, $F \in VE$. ($H_e E \overset{\text{wrt}}{\cong} T_{\pi(e)} V$)

$\tilde{\alpha}(Y) = \beta(X, F) + \alpha(X)$

If $Y \in T_v E$ for $v \in V$, then $\tilde{\alpha}(Y) = \alpha(X)$ $\Rightarrow \tilde{\xi}_v = \ker \tilde{\alpha}_v = \xi_v \oplus E_v$
$\because \beta$ vanishes

Lastly, $d\tilde{\alpha}|_{E_v}(F, F') = \eta_v((0,F), (0,F')) = \omega_{E_v}(F, F')$ for $F, F' \in E_v \cong V_v E$

An immediate corollary of the h-principle we proved is:

Cor: Let (V, ξ_V) & (W, ξ_W) be contact manifolds, let $A \subset V$ be a polyhedron of positive codimension. Then the local h-principle holds for isocontact immersions (also other forms)

$$(\mathcal{O}_p A, \xi_V|_{\mathcal{O}_p A}) \to (W, \xi_W)$$

Pf: $\mathcal{R}_{isocont}$ is invariant wrt $\mathrm{Diff}_{cont}(V)$ which is capacious.

From the previous talk, $\mathcal{R}_{isocont}$ is locally integrable & microflexible.

Thm: If $\dim V < \dim W$, then (all forms of) the h-principle hold for isocontact immersions
(Gromov) $(V, \xi_V) \to (W, \xi_W)$

We prove the non-parametric cases throughout.

Recall $\mathcal{R}_{isocont} = \{(v,w,\Phi) \in J^1(V,W) \mid \Phi: T_vV \to T_wW \text{ mono}, \Phi^{-1}(\xi_W) = \xi_V, \Phi: \xi_V \to \xi_W \text{ sympl}\}$

Pf: A formal solution is det'd by $F: TV \to TW$ a bundle monomorphism st $F^{-1}(\xi_W) = \xi_V$ and $F: \xi_V \to \xi_W$ is a symplectomorphism levelwise.

$\therefore F(\xi_V) \subset \xi_W$ are symplectic subspaces wrt $CS(\xi_W)$. Let $\omega_W \in CS(\xi_W)$, $\omega_V \in CS(\xi_V)$

consists of such sympl structures

Note: If $\xi = \ker \alpha$ locally, $d\alpha|_\xi$ is a sympl form on ξ det'd upto mult by a fn.

So are $(F(\xi_V))^{\omega_W} \subset \xi_W$. Let $N \xrightarrow{\pi} V$ be the bundle consisting of these fibers, ie the fiber N_v over $v \in V$ is $(F(\xi_{V_v}))^{\omega_W}$

This is a symplectic VB over a contact mfd (V, ξ_V) & Lemma 2 applies giving a contact str $\tilde{\xi} =: \xi_N$ in a nbhd $\mathcal{O}_p V$ of V in N st $(\xi_N)_v = \xi_{V_v} \oplus N_v$ and st ω_W on the fibers is in $CS(\xi_N)$. ($\Rightarrow (V, \xi_V)$ is a contact submfd of $(\mathcal{O}_p V, \xi_N)$)

From the proof of Lemma 2, $\omega_N := \eta + \pi^* \omega_V$ is a conformally sympl str on ξ_N

Claim: $(F(\xi_{V_v}))^{\omega_W} = (\xi_{V_v})^{\omega_N}$ viewing $\xi_{V_v} \subset \xi_{N_v}$

Pf: We prove the inclusion $\subseteq$. The claim then follows as they are both of the same dimension.

Suppose $\tilde{w} \in (F(\xi_{V_v}))^{\omega_W}$. Then
Let $u \in \xi_{V_v}$.

$$\omega_N(\tilde{w}, u) = (\eta + \pi^*\omega_V)(\tilde{w}, u)$$
$$= \eta(\tilde{w}, u) \qquad \because \pi_* \tilde{w} = 0 \text{ in a fiber}$$
$$= 0 \qquad \because u \in \xi_{V_v} \subset T V_v$$

$\Rightarrow \tilde{w} \in (\xi_{V_v})^{\omega_N}$

Hence the fibers of $N \to V$, which are sympl complements of $F(\xi_V)$ in W are also exactly the symplectic complements of ξ_V in $\mathcal{O}_p V \subset N$. $\Rightarrow$ F can be extended to an isocontact (Use the above to extend F to $TN|_V$ and then extend it over a neighbourhood) homomorphism $\tilde{F}: (T(\mathcal{O}_p V), \xi_N) \to (TW, \xi_W)$. Since these are of the same dimension and (see Talk 4) $\mathcal{O}_p V$ is open, $\mathcal{O}_p V$ has a core of positive codimension & the corollary above gives a htpy to $\tilde{F}: (T(\mathcal{O}_p V), \xi_N) \to (TW, \xi_W)$ st $\tilde{F} = T\tilde{f}$ for $\tilde{f}: (\mathcal{O}_p V, \xi_N) \to (W, \xi_W)$ isocontact immersion $\Rightarrow f := \tilde{f}|_V : (V, \xi_V) \to (W, \xi_W)$ is also an isocontact immersion.

Thm: (Gromov, Duchamp) The h-principle holds for Legendrian immersions $V \to (W, \xi_W)$ if $\dim W = 2\dim V + 1$

Pf: Recall $\mathcal{R}_{Leg} = \{ (v, w, \Phi) \mid \Phi : T_vV \to T_wW$ monomorphism, $\Phi(T_vV) \subset \xi_W$ as a Lagrangian subspace wrt $CS(\xi_W) \}$
$\in J^1(V, W)$

Let $F : TV \to TW$ be (det'd by) a formal solution.

For each $v \in V$, $F : T_vV \to T_wW$ ($w = bs\, F(v)$) maps T_vV into a Lagrangian subspace of ξ_W (isotropic)

The idea will be to show that F extends to an isocontact immersion

$\hat{F} : T(T^*V \times \mathbb{R}) \to TW$ where $T^*V \times \mathbb{R}$ has its contact str coming from $dz - \lambda_{can}$

$\cong T(T^*V) \times \mathbb{R}$

Recall that $T_{(q,p)}(T^*V) \cong T_qV \oplus T_q^*V$ by choosing a connection on $T^*V \to V$

$T_vV \subset T_{(v,p)}(T^*V)$ is an isotropic subspace & so is $(T_v^*V) \subset T_{(v,p)}(T^*V)$

$\therefore$ We should map T_v^*V into a complementary isotropic subspace of ξ_W

I don't have a rigorous proof for this. The following are some approaches we could use I think:

1) Note: For L a Lagrangian subspace of ξ, $\xi|L \to L^* : [w] \mapsto (\ell \mapsto \omega_W(-, w))$ is an isomorphism. $\therefore$ Given $\phi \in L^*$, $\exists ! [\tilde{w}] \in \xi|L$ st $\omega_W(v, \tilde{w}) = \phi(v)$ $\forall v \in L$.
So take $L = F(T_vV)$ and construct a linear map $T_v^*V \xrightarrow{F'} \xi_W$ st $\omega_W(F(v), F'(\phi)) = \phi(v)$
st the image of F' is an isotropic subspace

2) Since $F(T_vV)$ is a Lagrangian subspace of ξ_W, there is a symplectomorphism of ξ_W taking $F(T_vV)$ to $\langle x^1, .., x^n \rangle \subseteq \xi_W$ in Darboux coordinates of ξ_W. Then appropriately map T_v^*V to the inverse image of $\langle y^1, .., y^n \rangle \subseteq \xi_W$ under the symplectomorphism.
Problem: This must be done in a smooth way, but the symplectomorphism is not canonical

3) Use Weinstein's Neighbourhood theorem for the isotr. injection $F(T_vV) \to \xi_W$. Again, will have to ensure smoothness.

$\rightsquigarrow$ We get $\hat{F} : T(T^*V \times \mathbb{R}) \to TW$ st

st $\hat{F}^* \xi_W$ is the contact structure of $T^*V \times \mathbb{R}$

So $\hat{F}$ is an isocontact homomorphism of equidimensional mfds.

As before, the local h-principle for isocontact immersions implies that $\hat{F}$ is htpic to $\tilde{F} = T\tilde{f}$ where $\tilde{f} : T^*V \times \mathbb{R} \to W$ is an isocontact immersion.

$f := \tilde{f}|_V : V \to W$ is then a Legendrian immersion.

Lagrangian immersions

Let $(W, d\alpha = \omega)$ be an exact symplectic mfd of dim $2n$. Let $\dim V = n$.

A Lagrangian immersion $f : V \to (W, d\alpha = \omega)$ is called exact if $f^*\alpha$ is exact.

Prop: Let V be n-diml & W be $2n$-diml. An isotropic monomorphism $F : TV \to TW$ (ie F monomorphism st $F^*\omega = 0$) is homotopic in $\mathcal{R}_{\text{isot}}$ to $\tilde{F} : TV \to TW$ st $\tilde{F} = Tf$ for $f : V \to (W, \omega = d\alpha)$ an exact Lagrangian immersion

Pf: Observe the following - (Here we are denoting a contact manifold by (W, λ) where λ is a contact form on W)

1) $(W, \omega = d\alpha)$ sympl mfd $\Rightarrow$ $(W \times \mathbb{R}, \lambda = dt - \alpha)$ is a contact mfd

2) $f : V \to (W, \omega = d\alpha)$ is an exact Lagrangian immersion w/ $f^*\alpha = dg$, $g \in C^\infty(V)$

$\Rightarrow$ $\tilde{f} : V \to (W \times \mathbb{R}, \lambda)$ defined by $\tilde{f}(v) = (f(v), g(v))$ is a Legendrian immersion

$[\lambda(T\tilde{f}(X)) = (dt - \alpha)(Tf(X), Tg(X)) = -\alpha(Tf(X)) + dg(X) = (dg - f^*\alpha)(X) = 0]$

3) $\tilde{f} : V \to (W \times \mathbb{R}, \underset{dt-\alpha}{\lambda})$ w/ $\tilde{f} = (f, g)$ is a Legendrian immersion

$\Rightarrow$ $f : V \to (W, d\alpha)$ is an exact Lagrangian immersion.

$[\lambda(T\tilde{f}(X)) = (dt - \alpha)(Tf(X), Tg(X)) = 0 \;\; \forall X \in TV$ as $\tilde{f}$ is Legendrian

$\Rightarrow dg(X) = \alpha(Tf(X)) \;\; \forall X \in TV$

$\Rightarrow f^*\alpha = g$

Also, $Tf(X) = 0 \Rightarrow dg(X) = \alpha(Tf(X)) = 0 \Rightarrow T\tilde{f}(X) = 0 \Rightarrow X = 0$]

4) On the level of formal data, $F : TV \to (TW, d\alpha)$ is an isotropic mono

$\Rightarrow$ $\tilde{F} : TV \to (T(W \times \mathbb{R}), \lambda)$ defined as $\tilde{F}(X) = (F(X), \alpha(F(X)))$ is an isotropic mono.

Given $F : TV \to TW$ an isotropic mono, get $\tilde{F}$ as above; $\tilde{F}$ is a formal Legendrian immersion, so by the h-principle for $\mathcal{R}_{Leg}$, $\tilde{F}$ is homotopic to $\tilde{F}' = T\tilde{f}'$ for a Legendrian immersion. By 3), $f = pr_W \circ \tilde{f}'$ is an exact Lagrangian immersion w/ $F' := Tf = pr_{TW} \circ \tilde{F}'$. Projections of Legendrian subspaces of $T_w W \times \mathbb{R}$ are Lagrangian subspaces of $T_w W$, so we get a homotopy from F to $F' = Tf$.

Q: What about Lagrangian immersions into general symplectic mfds?

Suppose $F_t : TV \to (TW, \omega)$ is a htpy of formal Lagrangian immersions.

Then $[(bs F_0)^* \omega] = [(bs F_1)^* \omega] \in H^2_{dR}(V)$ & if F_1 is the differential of a Lagrangian immersion ie $F_1 = T(bs F_1)$, then $(bs F_1)^* \omega = 0$

$\therefore$ F_0 can be htpic to the differential of a Lagrangian immersion only if

$[(bs F_0)^* \omega] = 0 \in H^2_{dR}(V)$.

It turns out that this necessary condition is also sufficient.

Thm: The h-principle holds for Lagrangian immersions if the formal solutions F are (Gromov, Lees) also assumed to satisfy $[(bs(F))^* \omega] = 0 \in H^2_{dR}(V)$

I haven't resolved the problem yet.

Isosymplectic immersions

Let (V, ω_V), (W, ω_W) be symplectic mfds with $\dim W \geq \dim V$ and $A \subset V$ be a subpolyhedron of positive codim

Lemma: Let $U \subset V \times W$ be open st $\Omega = \omega_V \oplus \omega_W$ is exact on U. Then the h-principle holds for Lagrangian / isotropic sections $\mathcal{O}p A \to U \subset V \times W$

Idea of the pf: Say $\Omega|_U = d\alpha$, $\alpha \in \Omega^1(U)$. Convert this problem to a problem about Legendrian/isotropic sections $\mathcal{O}p A \to (U \times \mathbb{R}, \lambda = dz - \alpha)$ & prove that this differential rel is locally integrable, microflexible & invariant wrt a small nbhd of identity in $\mathrm{Ham}(V, \omega_V)$.

I need to think more about this step

For $A \subset V$, denote $\mathrm{Sec}^0_{\mathcal{O}p A} \mathcal{R}_{isosymp} = \{F \in \mathrm{Sec}_{\mathcal{O}p A} \mathcal{R}_{isosymp} \mid [(bsF)^* \omega_W] = [\omega_V|_{\mathcal{O}p A}] \in H^2_{dR}(\mathcal{O}p A)\}$

Thm: The local h-principle holds for the inclusion $\mathrm{Hol}_{\mathcal{O}p A} \mathcal{R}_{isosymp} \to \mathrm{Sec}^0_{\mathcal{O}p A} \mathcal{R}_{isosymp}$

$F : T(\mathcal{O}p A) \to TW$, $F^* \omega_W = \omega_V$, $[(bsF)^* \omega_W] = [\omega_V|_{\mathcal{O}p A}]$

Pf: Let $F \in \mathrm{Sec}^0_{\mathcal{O}p A} \mathcal{R}_{isosymp}$, $f := bsF : \mathcal{O}p A \to W$ and let G be the 0-jet part of F, ie the graph $\{\hat{f}(x) = (x, f(x)) \mid x \in \mathcal{O}p A\} \subset (V \times W, \Omega := \omega_V \oplus -\omega_W)$.

A tangent vector of G is of the form $(X, Tf(X)) \in TV \times TW$ for $X \in TV$

$\Omega|_G((X, Tf(X)), (Y, Tf(Y))) = (\omega_V - f^* \omega_W)(X, Y) = d\alpha(X, Y) = d(pr_V^* \alpha)((X, Tf(X)), (Y, Tf(Y)))$; $pr_V^* \alpha \in \Omega^1(G)$

$\therefore$ $\Omega|_G$ is exact.

$\Rightarrow$ Ω is exact on a nbhd U (by retracting a nbhd of the graph to the graph)

Define $\hat{F} : T(\mathcal{O}p A) \longrightarrow (TV \times TW, \omega_V \oplus -\omega_W)$ by

$$X \longmapsto (X, F(X))$$

$\hat{F}^*(\omega_V \oplus -\omega_W)(X) = (\omega_V \oplus -\omega_W)(X, F(X)) = \omega_V(X) - \omega_W(F(X)) = 0$ as $F^*\omega_W = \omega_V$ by hyp.

Obs: $bs\,\hat{F} = \hat{f} : \mathcal{O}p A \to V \times W$. In fact, $\hat{f} : \mathcal{O}p A \to U \subset V \times W$.

So $\hat{F}$ is a formal isotropic section $T(\mathcal{O}p A) \to TU \subset TV \times TW$ & Ω is exact on U.

ie $\hat{F} \in \mathrm{Sec}_{\mathcal{O}p A} \tilde{\mathcal{R}}_{isotr}$ $\tilde{\mathcal{R}}_{isotr} \subset (\mathcal{O}p A \times V \times W)^{(1)}$, the PDR for isotropic immersions of $\mathcal{O}p A$ into $V \times W$

The lemma above implies that $\hat{F}$ is homotopic through formal isotropic sections to $\hat{\hat{F}} = T\hat{\hat{f}} \in \mathrm{Hol}_{\mathcal{O}p A} \tilde{\mathcal{R}}_{isotr}$, $\hat{\hat{f}}$ is C^0-close to $\hat{f}$ & $\hat{\hat{f}}$ is an isotropic section $\mathcal{O}p A \to U \subseteq V \times W$.

Note: For $p \in \mathcal{O}p A$ & $X, Y \in T_pV$, $\Omega(T\hat{f}(X), T\hat{f}(Y)) = (\omega_V \oplus -\omega_W)((X, Tf(X)), (Y, Tf(Y)))$

$$= \omega_V(X,Y) - \omega_W(Tf(X), Tf(Y))$$
$$= (\omega_V - f^*\omega_W)(X,Y)$$

$\therefore$ f is an isosymplectic immersion $\Leftrightarrow$ $\hat{f}$ is an isotropic section

Say $\hat{\hat{f}} = (\mathrm{Id}, \tilde{f})$. Then $\tilde{f}$ is an isosymplectic immersion & F is homotopic to $T\tilde{f}$ via formal isosymplectic immersions (by composing the obtained htpy w/ the projection to TW) & $\tilde{f}$ is C^0-close to f.

Thm: Let $(V, \omega_V), (W, \omega_W)$ be symplectic mfds w/ $\dim W > \dim V$.
(Gromov) Then the h-principle holds for isosymplectic immersions $(V, \omega_V) \to (W, \omega_W)$ if the formal solutions F are assumed to also satisfy $[(bs F)^*\omega_W] = [\omega_V]$

Pf: This is similar to the proof of h-principle for isocontact immersions. Given $F : (TV, \omega_V) \to (TW, \omega_W)$ a formal solution, let $N \xrightarrow{\pi} V$ be the normal bundle to $F(V)$ in TW, ie $N_v = (F(T_vV))^{\omega_W}$. By Lemma 1, $\exists$ a symplectic str ω_N on a nbhd $\mathcal{O}p V$ in N st $\omega_N|_V = \omega_V$ & $\omega_N|_{N_v} = \omega_W$ on fibers.

As before, we deduce $(F(T_vV))^{\omega_W} = (T_vV)^{\omega_N}$, ie the fibers of N are symplectic complements of $TV \subset TN|_V$. Then we can extend F to

$$\hat{F} : (T(\mathcal{O}p V), \omega_N) \to (TW, \omega_W),$$ an equidim^l isosymplectic immersion. $\mathcal{O}p V$ is open, thus has a core of positive codim. The additional assumption on F ensures $[(bs\hat{F})^*\omega_W] = [\omega_N|_{\mathcal{O}p V}]$, so the previous theorem applies to give a htpy of $\hat{F}$ to $\tilde{F} = T\tilde{f}$ for $\tilde{f} : (\mathcal{O}p V, \omega_N) \to (W, \omega_W)$ an isosymplectic immersion. Restricting to the zero section, we get $f = \tilde{f}|_V : (V, \omega_V) \to (W, \omega_W)$ which is also isosymplectic as $V \hookrightarrow \mathcal{O}p V$ is an isosympl immersion.