Motivation

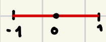
- Given an open differential relation $R \subseteq J^1(\mathbb{R}, \mathbb{R}^q) = \underset{\text{"time"}}{\mathbb{R}} \times \underset{\text{"position"}}{\mathbb{R}^q} \times \underset{\text{"velocities"}}{\mathbb{R}^q}$ (i.e. an open subset) and a formal solution $F = (f, \ell) : I \longrightarrow R$, $t \longmapsto (t, f(t), \ell(t))$ smooth but suppose extend R in the sense that we allow for "more velocities at time t and position $f(t)$". Moreover, suppose that the extension of R now yields that f is a genuine solution (So extension of R captures $\dot{f}(t)$), Can we get a genuine solution of R?

- The answer is YES. We can even find a homotopy, which connects F with a genuine sol. $F_1 = (f_1, \dot{f}_1)$ and f and f_1 are C^0-close together.

- To do this, we use a technique called "convex integration". In the following, I want to show you how the main-argument for the proof of the above statement works (even if some maps below are not smooth):

- Regard $f \equiv 0$ and suppose there are velocities $a_1, \dots, a_n$ in R s.t. $0 \in \operatorname{int}(\operatorname{Conv}(a_1, \dots, a_n))$. Then we decompose $0 = \sum \alpha_i a_i$ and α_i gives us the time we spend with velocity a_i. By defining a path, which always goes between the a_i in high frequence and integrating it gives us a "solution", which is C^0-close to 0.

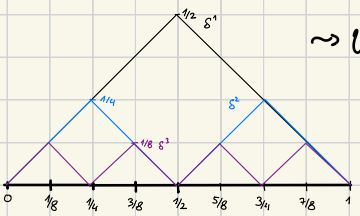
Example:

$a_1 = -1$, $a_2 = 1$ "possible velocities" and $\frac{1}{2}a_1 + \frac{1}{2}a_2 = 0$ 

$\rightsquigarrow \delta^1(t) = \begin{cases} 1 & \text{for } t \in [0, \frac{1}{2}) \\ -1 & \text{for } t \in [\frac{1}{2}, 1] \end{cases} \rightsquigarrow \int_0^t \delta^1(\sigma)\,d\sigma = \begin{cases} t & \text{if } t \in [0, \frac{1}{2}) \\ 1-t & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$

↳ right velocities but is not C^0-close to 0

- Make it oscilate fast between 1 and -1 : $\delta^2(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{1}{4}) \\ -1 & \text{if } t \in [\frac{1}{4}, \frac{1}{2}) \\ 1 & \text{if } t \in [\frac{1}{2}, \frac{3}{4}) \\ -1 & \text{if } t \in [\frac{3}{4}, 1] \end{cases} \rightsquigarrow \int_0^t \delta^2(\sigma)\,d\sigma = \begin{cases} t & \text{if } t \in [0, \frac{1}{4}) \\ \frac{1}{2} - t & \text{if } t \in [\frac{1}{4}, \frac{1}{2}) \\ t - \frac{1}{4} & \text{if } t \in [\frac{1}{2}, \frac{3}{4}) \\ \frac{5}{4} - t & \text{if } t \in [\frac{3}{4}, 1] \end{cases} \rightsquigarrow \dots$



$\rightsquigarrow$ Will C^0-approx. 0.

Rmk:

- "Map", "path" and "section" will always mean that the functions are C^∞.

Important Definitions

Interpretation of differential relations as "differential inclusions"

- Let $R \subseteq J^1(\mathbb{R},\mathbb{R}^q)$ be a differential relation and let $(t,y,\dot{y}) \in J^1(\mathbb{R},\mathbb{R}^q)$. We define

$\Omega(t,y) := R \cap \overset{\text{Can be empty}}{P_{t,y}} := R \cap \overset{\{t\}\times\{y\}\times\mathbb{R}^q \,=}{(p_0^1)^{-1}(t,y)}$, where $p_0^1 : \underset{= \mathbb{R}\times\mathbb{R}^q\times\mathbb{R}^q}{J^1(\mathbb{R},\mathbb{R}^q)} \longrightarrow \underset{= \mathbb{R}\times\mathbb{R}^q}{J^0(\mathbb{R},\mathbb{R}^q)}$, $(\tilde{t},\tilde{y},\dot{\tilde{y}}) \mapsto (\tilde{t},\tilde{y})$. Moreover, we

naturally identify $\Omega(t,y) \simeq \{\dot{y} \in \mathbb{R}^q \mid (t,y,\dot{y}) \in R\}$ "Space of possible velocities at y" and get that $(t,y,\dot{y}) \in R \iff \dot{y} \in \Omega(t,y)$.

Def^n 1: "Modifying Velocities"

- Given a differential relation $R \subseteq J^1(\mathbb{R},\mathbb{R}^q)$ and a Section $F = (f,\ell) : \mathbb{R} \longrightarrow R$ i.e. Smooth maps $f:\mathbb{R}\to\mathbb{R}^q$, $\ell:\mathbb{R}\to\mathbb{R}^q$ s.t. $(t,f(t),\ell(t)) \in R$ $\forall t \in \mathbb{R}$.

i) $\mathrm{Conn}_{F(t)}(R)$ will denote the path component of $\Omega((t,f(t)))$ which contains $F(t) \subseteq \{t\}\times\{f(t)\}\times\mathbb{R}^q$, but we can always identify with subset of $\mathbb{R}^q$ as above

ii) $\mathrm{Conv}_{F(t)}(R) := \mathrm{Conv}\left(\mathrm{Conn}_{F(t)}(R)\right) \subseteq P_{t,f(t)} = \{t\}\times\{f(t)\}\times\mathbb{R}^q$

iii) $\mathrm{Conv}_F(R) := \bigcup_{t\in\mathbb{R}} \mathrm{Conv}_{F(t)}(R) \subseteq \mathbb{R}\times\mathbb{R}^q\times\mathbb{R}^q = J^1(\mathbb{R},\mathbb{R}^q)$ $\rightsquigarrow$ This is differential relation

iv) Suppose F is a formal solution of R. We will call F a short formal solution, if $f:\mathbb{R}\longrightarrow\mathbb{R}^q$ is a genuine solution of $\mathrm{Conv}_F R$ i.e. $(t,f(t),\dot{f}(t)) \in \mathrm{Conv}_F(R)$ $\forall t\in\mathbb{R}$.

v) We will call R fiberwise path-connected if $\Omega(t,y)$ is path-connected $\forall (t,y) \in \mathbb{R}\times\mathbb{R}^q$

vi) $R \subseteq J^1(\mathbb{R},\mathbb{R}^q)$ is called ample if $\forall (t,y) \in \mathbb{R}\times\mathbb{R}^q$: $\mathrm{Conv}(\overset{\text{path-connected component, where } \dot{y} \text{ lies}}{\mathrm{Conn}_{\dot{y}}}(\Omega(t,y))) = \mathbb{R}^q$ $\forall \dot{y} \in \Omega(t,y)$ (if $\Omega(t,y)=\emptyset$, then ok.)

Main Lemma and Proof Idea

Lemma 1: One-dimensional convex integration (25.3.1)

WLoG $R \subseteq I \times \mathbb{R}^q \times \mathbb{R}^q$ open.

· Let $R \subseteq J^1(\mathbb{R}, \mathbb{R}^q)$ be an open differential relation and $F = (f, \varphi) : I \longrightarrow R$ be a short formal solution of R.

· Then there exists a cont. map $H : I \times I \longrightarrow R$, $(\tau, t) \longmapsto F_\tau(t) := (f_\tau(t), \varphi_\tau(t))$

i) $\forall \tau \in [0,1]$: $H(\cdot, \tau)$ is a short formal solution and f_τ is (arbitrarily) C^0-close to f.

ii) $H(\cdot, 0) = F$, $H(\cdot, 1)$ is a genuine solution of R and $H(0, \tau) = F(0)$, $H(1, \tau) = F(1)$ $\forall \tau \in [0,1]$.

"Homotope short formal sol. along short formal sol. to a genuine sol."

Rmk:

· If the formal solution F is already genuine in near ∂I i.e. $(f, \varphi) = (f, f') \in R$ in a nbhd of 0 and 1 then the above homotopy H can be choosen fixed near ∂I: Apply Lemma 1 to $[\delta, 1-\delta] \subset I$ and get $\tilde{H} : [\delta, 1-\delta] \times I \longrightarrow R$. Now define

$$H : I \times I \longrightarrow R, \quad (t, \tau) \longmapsto \begin{cases} F(t) & \text{if } t \notin [\delta, 1-\delta] \\ \tilde{H}(t, \tau) & \text{else} \end{cases}$$

. This map is cont. b.c. by ii), we have $\tilde{H}(\delta, \tau) = F(\delta)$ and $\tilde{H}(1-\delta, \tau) = F(1-\delta)$ $\forall \tau \in [0,1]$.

Problem: The maps $H(\cdot, \tau)$ are not necessarily differentiable at the "gluing points" δ and $1-\delta$ but we can fix this by using openness of R and working in small nbhds $(\delta-\varepsilon, \delta+\varepsilon)$ and $(1-\delta-\varepsilon, 1-\delta+\varepsilon)$, where we use a smooth interpolation which is small enough, so we stay in R and C^0-close to f.

Proof Idea:

① Reduce to the case when $f \equiv 0$

② Localize the problem and use the "good local form" of R

③ Prove the lemma in the easier setting that we got from localization

④ Glue local solutions to get desired statement.

Proof of the Main Lemma

Lemma 2: "Reduction to $f \equiv 0$"

· Suppose we have proven lemma 1 in the case where $f \equiv 0$. Then it holds in general.

Proof:

· We define $\tilde{R} := \{ (t, z, \dot{z}) \in I \times \mathbb{R}^q \times \mathbb{R}^q \mid \overbrace{(t, z + f(t), \dot{z} + \dot{f}(t))}^{\psi(t,z,\dot{z})} \in R \}$ "Variation relation along f"

· Then $\tilde{R}$ is open b.c. $\tilde{R} = \psi^{-1}(R)$ and $\psi : I \times \mathbb{R}^q \times \mathbb{R}^q \longrightarrow I \times \mathbb{R}^q \times \mathbb{R}^q$ is cont.

· Moreover $\tilde{F} = (0, \varphi - \dot{f}) : I \longrightarrow \tilde{R}$ is a short formal solution b.c. $0 \in \mathrm{Conv}_{\tilde{F}} \tilde{R}$. So we can apply lemma 1, get $\tilde{H} : I \times I \longrightarrow \tilde{R}$ with

$\tilde{H}(t, \tau) = (t, \tilde{f}_\tau(t), \dot{\tilde{f}}_\tau)$. We define $H : I \times I \longrightarrow R, (t, \tau) \longmapsto (t, \tilde{f}_\tau(t) + f(t), \dot{\tilde{f}}_\tau(t) + \dot{f}(t))$, which yields the desired homotopy.

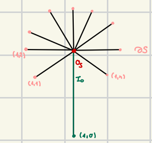
$\square$

Def^n 2: (Abstract) Flowers

i) Let $n \in \mathbb{N}$ and $(I_0, 0), (I_1, 0), \dots, (I_n, 0)$, where $I_i = [0,1]$ $\forall i = 0, \dots, n$. We will call $(S, 0_S) := I_0 \vee I_1 \vee \dots \vee I_n$

an abstract flower. The interval $I_0 \subseteq S$ is called the stem of the flower, all other intervalls $I_i \subseteq S$, $i = 1, \dots, n$

are called the petals. $\partial S := \{ \underbrace{(1,1)}_{1 \text{ in } I_1 \subseteq S}, \underbrace{(1,2)}_{1 \text{ in } I_2 \subseteq S}, \dots, \underbrace{(1,n)}_{1 \text{ in } I_n \subseteq S} \}$ ⇝ the union of free ends of the petals $I_1, \dots, I_n$ of the flower S.



(Ordering not important)

ii) A map (Smooth means smoothness of parametrization maps; ↳ Not necessarily bijective) $\psi : S \longrightarrow \mathbb{R}^q$, and sometimes also its image $\Psi := \psi(S)$, will be called a flower.

· Note that giving a map $\psi : S \longrightarrow \mathbb{R}^q$ is equivalent to giving (smooth) paths $\psi_i : I \longrightarrow \mathbb{R}^q$ for $i = 0, \dots, n$ where $\psi_0(0) = \psi_1(0) = \dots = \psi_n(0)$.

· For a flower $\Psi = \psi(S)$, we set $a_i := \psi_i(1)$ $i = 1, \dots, n$ and $\partial \Psi := \psi(\partial S) = \{ \psi_1(1), \dots, \psi_n(1) \}$.

Lemma 3: "Localization of the Problem" (25.4.2)

· Let $R \subseteq I \times \mathbb{R}^q \times \mathbb{R}^q$ be an open differential relation and $F = (0, \varphi) : I \longrightarrow R$ be a short formal solution.

· Then $\exists \delta > 0$ s.t. $\forall t_0 \in [0, 1-\delta]$ one can choose a flower $\Psi = \psi(S) \subseteq P_{t_0, 0} = \{t_0\} \times \{0\} \times \mathbb{R}^q$ s.t.

i) $0 \in \mathrm{int}(\mathrm{Conv}(\partial \Psi))$

ii) $\psi_0(t) = \varphi(t_0 + \delta t)$, $t \in I$ "Stem is given by φ"

iii) $[t_0, t_0 + \delta] \times \mathbb{D}^q_\varepsilon \times \Psi \subset R$ for sufficiently small $\varepsilon > 0$.

(disk around 0 in $\mathbb{R}^q$ of radius ε; also uniform for all $t_0 \in [0, 1-\delta]$)

Proof:

- Let $t_0 \in I$ be arb. Since $(t_0, 0, 0) \in \mathrm{Conv}_{F(t_0)} R = \mathrm{Conv}(\mathrm{Conn}_{F(t_0)} R)$ $\exists n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in (0,1]$ with $\sum_{i=1}^n \lambda_i = 1$ and $a_1, \dots, a_n \in \mathrm{Conn}_{F(t_0)}(R)$ s.t. $0 = \sum_{i=1}^n \lambda_i a_i \Rightarrow 0 \in \mathrm{Conv}\{a_1, \dots, a_n\}$. Now there are two cases:

Case 1: $0 \in \mathrm{int}(\mathrm{Conv}\{a_1, \dots, a_n\})$ i.e. $\lambda_i \in (0,1)$ $\forall i$

- Then since $f(t_0), a_1, \dots, a_n \in \mathrm{Conn}_{F(t_0)}(R)$ and by path connectedness of this space $\exists \psi_i : I \longrightarrow \mathrm{Conn}_{F(t_0)}(R)$ with $\psi_i(0) = f(t_0)$ and $\psi_i(1) = a_i$. (smooth paths b.c. R open $\Rightarrow \Omega(t, f(t))$ open $\Rightarrow \mathrm{Conn}_{F(t_0)}(R)$ open) These paths will be the petals of our flower Ψ while the path $\psi_0(t) := f(t_0 + \delta t)$ is its stem.
- By openness of R, we get that for all $t_i \in [0,1]$ $\exists \delta_i > 0$ and $\varepsilon_i > 0$ s.t. $(t_i - \delta_i, t_i + \delta_i) \cap I \times D^q_{\varepsilon_i} \times \Psi \subseteq R$. Can cover I by $(t_i - \delta_i, t_i + \delta_i) \cap I \rightsquigarrow$ Only need fin. many $\rightsquigarrow$ Define $\delta := \min_{i \in J} \frac{\delta_i}{2}$. For any $t_0 \in [0, 1-\delta]$ $\exists i \in J : [t_0, t_0 + \delta] \subseteq (t_i - \delta_i, t_i + \delta_i) \cap I$ and hence for $\varepsilon := \min_{i \in J} \varepsilon_i / 2 : [t_0, t_0 + \delta] \times D^q_\varepsilon \times \Psi \subseteq R$.

Case 2: $\exists i = 1, \dots, n : 0 = a_i$ i.e. $\lambda_i = 1$.

- Then by openness of R and by local path-connectedness of $\mathbb{R}^q$, $\mathrm{Conn}_{F(t_0)}(R) \subseteq \mathbb{R}^q$ is open. Then $\exists r > 0 : B_r(0) \subseteq \mathrm{Conn}_{F(t_0)}(R)$
- For $a_1 = \frac{r}{2} e_1, \dots, a_{q+1} = -\frac{r}{2q} \sum_{i=1}^q e_i$ we get that $0 = \frac{1}{q+1} \sum_{i=1}^{q+1} a_i \Rightarrow 0 \in \mathrm{int}(\mathrm{Conv}(a_1, \dots, a_{q+1}))$. Return to Case 1.

$\square$

- In order to prove Lemma 1 in the local case, we need the following concept:

Defn 3: "(weighted) Product of paths and balanced paths"

- Let $K \in \mathbb{N}$, $p_1, \dots, p_K : I \longrightarrow \mathbb{R}^q$ be paths and $\alpha_1, \dots, \alpha_K \in (0,1]$ (α_i can be 0 if p_i is constant) s.t. $\alpha_1 + \dots + \alpha_K = 1$.
- We define $p := p_1 \cdot p_2 \cdot \ldots \cdot p_K : I \longrightarrow \mathbb{R}^q$, $t \longmapsto p_i\left(\frac{t - t_{i-1}}{\alpha_i}\right)$ for $t \in (t_{i-1}, t_i]$ and $p(0) := p_1(0)$ where $t_i := \alpha_1 + \dots + \alpha_i$ for $i = 1, \dots, K$, $t_0 := 0$.
  (a priori not cont.)
- We call p the weighted product of paths (not necessarily continuous) and if $\alpha_i = 1/K$ $\forall i = 1, \dots, K$, we call p the uniform product of paths.
- Given a path $p : I \longrightarrow \mathbb{R}^q$, we will denote by p^N the uniform product $p \cdot \ldots \cdot p$ of N factors and define $p^{-1} : I \longrightarrow \mathbb{R}^q$, $t \longmapsto p(1-t)$.
- Moreover, we will define $\int p(\sigma) d\sigma : I \longrightarrow \mathbb{R}^q$, $t \longmapsto \int_0^t p(\sigma) d\sigma$ (integrate componentwise) and we will call p balanced if $\int_0^1 p(\sigma) d\sigma = 0$.

Property 1: "Multiplicativity of the integral" (25.4.3)

- Let $p_i : I \longrightarrow \mathbb{R}^q$, $i = 1, \dots, N$ be balanced paths. Then $\int (p_1 \cdot \ldots \cdot p_N)(\sigma) d\sigma = 1/N \left(\int p_1(\sigma) d\sigma \cdot \ldots \cdot \int p_N(\sigma) d\sigma \right)$ where the products on both sides are uniform.
- This shows in particular that $p_1 \cdot \ldots \cdot p_N$ is also balanced.

Proof:

- Let $t \in I$ be arb. Then $\exists i \in \{1,\dots,N\}$ s.t. $t \in (\frac{i-1}{N}, \frac{i}{N}]$ and with this:

$$\int_0^t (p_1 \cdot \ldots \cdot p_N)(\sigma)\,d\sigma = \int_0^{1/N} p_1(\overbrace{N\sigma}^{\tau})\,d\sigma + \int_{1/N}^{2/N} p_2(\overbrace{N(\sigma - 1/N)}^{\tau})\,d\sigma + \ldots + \int_{\frac{i-1}{N}}^{t} p_i(\overbrace{N(\sigma - \tfrac{i-1}{N})}^{\tau})\,d\sigma$$

$$\overset{\text{Subsitution}\to}{=} 1/N \int_0^1 p_1(\tau)\,d\tau + 1/N \int_0^1 p_2(\tau)\,d\tau + \ldots + \frac{1}{N}\int_0^{N(t-\frac{i-1}{N})} p_i(\tau)\,d\tau$$

$$\overset{\text{balanced}\to}{=} \frac{1}{N}\int_0^{N(t-\frac{i-1}{N})} p_i(\tau)\,d\tau \underset{\substack{\uparrow \\ \text{Def}^n}}{=} 1/N \left(\int p_1(\tau)d\tau \cdot \ldots \cdot \int p_N(\tau)d\tau\right)(t).$$

$\square$

Property 2: "C^0-norm of uniform products" (25.4.4)

- Let $p_i : I \longrightarrow \mathbb{R}^q$, $i=1,\dots,N$ be balanced paths. Then $\| \int \overset{\text{uniform product}}{(p_1 \cdot \ldots \cdot p_N)}(\sigma)\,d\sigma \|_{C^0} = 1/N \max\{ \|\int p_1(\sigma)d\sigma\|_{C^0}, \dots, \|\int p_N(\sigma)d\sigma\|_{C^0} \}$.

Proof:

$$\| \int (p_1 \cdot \ldots \cdot p_N)(\sigma)\,d\sigma \|_{C^0} \overset{\text{Property 1}}{=} 1/N \, \| \int p_1(\sigma)d\sigma \cdot \ldots \cdot \int p_N(\sigma)d\sigma \|_{C^0} = 1/N \sup_{t \in [0,1]} \| (\int p_1(\sigma)d\sigma \cdot \ldots \cdot \int p_N(\sigma)d\sigma)(t) \|$$

$$= 1/N \max\left\{ \sup_{t\in[0,1/N]} \| \int_0^{Nt} p_1(\sigma)d\sigma \|, \sup_{t\in[1/N,2/N]} \| \int_0^{N(t-1/N)} p_2(\sigma)d\sigma \|, \dots, \sup_{t \in [\frac{N-1}{N},1]} \| \int_0^{N(t-\frac{N-1}{N})} p_N(\sigma)d\sigma \| \right\}$$

$$= 1/N \max\left\{ \sup_{t\in[0,1]} \| \int_0^{t} p_1(\sigma)d\sigma \|, \sup_{t\in[0,1]} \| \int_0^{1} p_2(\sigma)d\sigma \|, \dots, \sup_{t\in[0,1]} \| \int_0^{1} p_N(\sigma)d\sigma \| \right\}$$

$$= 1/N \max\left\{ \|\int p_1(\sigma)d\sigma\|_{C^0}, \|\int p_2(\sigma)d\sigma\|_{C^0}, \dots, \|\int p_N(\sigma)d\sigma\|_{C^0} \right\}$$

$\square$

Rmk:

Property 2 implies that if $p : I \longrightarrow \mathbb{R}^q$ is a balanced path, then $p^N \xrightarrow{N\to\infty} 0$ in C^0-norm.

Lemma 4: "Main-Lemma in local setting" (25.4.1)

- Let $\varepsilon > 0$ be arb., $\Psi = \Psi(S) \subset \mathbb{R}^q$ a flower with $0 \in \operatorname{int}(\operatorname{Conv} \partial \Psi)$ and $R = I \times D^q_\varepsilon \times \Psi$. Moreover, let

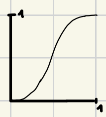
$F = (0, \varphi) : I \longrightarrow R$ be a formal solution of R (automatically short b.c. $0 \in \operatorname{Conv}(\Psi)$), where φ is the stem of Ψ.

Then there exists a cont. map $H : I \times I \longrightarrow R$ s.t.

i) $\forall \tau \in [0,1]$: $H(\cdot,\tau)$ is a short formal solution and (arbitrarily) C^0 close to 0.

ii) $H(\cdot,0) = F$, $H(\cdot,1)$ is a genuine solution of R and $H(0,\tau) = F(0)$, $H(1,\tau) = F(1)$ $\forall \tau \in [0,1]$.
↳ will ensure cont. when we "glue" later.

Proof:



- Let $\Psi = \{\varphi, \psi_1, \dots, \psi_k\}$ be the parametrizing maps for the flower Ψ. By a slight reparametrisation

we may assume that $\psi_i(t) \equiv \varphi(0)$ near $t=0$ and $\psi_i(t) \equiv a_i$ near $t=1$ for $i=1,\dots,k$ (to ensure smoothness)

• We consider the product $\psi = \psi_1 \cdot a_1 \cdot \psi_1^{-1} \cdot \ldots \cdot \psi_k \cdot a_k \cdot \psi_k^{-1}$, where the weights of constant paths a_i are $(1-\underset{\text{arb } \varrho>0}{\varrho})\underset{\alpha_i>0:\ \sum_{i=1}^{k}\alpha_i=1}{\alpha_i}$ and the weights of the other paths are $\varrho/2k$.

• We want to adjust α_i and ϱ s.t. ψ becomes balanced:

• For this, let
$$d := \int_0^{\varrho/2k} \psi_1\Big(\overbrace{\tfrac{2k\sigma}{\varrho}}^{\tau}\Big)d\sigma + \int_{\varrho/2k+(1-\varrho)\alpha_1}^{\varrho/2k+(1-\varrho)\alpha_1+\varrho/2k} \psi_1\Big(1-\underbrace{2k\Big(\tfrac{\sigma-\varrho/2k-(1-\varrho)\alpha_1}{\varrho}\Big)}_{\tau}\Big)d\sigma + \ldots + \int_{(2k-2)\varrho/2k+(1-\varrho)\sum_{i=1}^{k-1}\alpha_i}^{(2k-1)\varrho/2k+(1-\varrho)\sum_{i=1}^{k-1}\alpha_i} \psi_k\Big(\underbrace{2k\Big(\tfrac{\sigma-(2k-2)\varrho/2k-(1-\varrho)\sum_{i=1}^{k-1}\alpha_i}{\varrho}\Big)}_{\tau}\Big)d\sigma + \int_{(2k-1)\varrho/2k+(1-\varrho)}^{1} \psi_k\Big(1-\overbrace{2k\Big(\tfrac{\sigma-(2k-1)\varrho/2k-(1-\varrho)}{\varrho}\Big)}^{\tau}\Big)d\sigma$$
$$= \varrho/2k\int_0^1 \psi_1(\tau)d\tau \underset{\text{and - from changing integral bounds}}{\overset{\text{- from sub.}}{+}} \varrho/2k\int_0^1 \psi_1(\tau)d\tau + \ldots + \varrho/2k\int_0^1 \psi_k(\tau)d\tau + \varrho/2k\int_0^1 \psi_k(\tau)d\tau$$

$\Rightarrow \|d\| \leq \varrho/k \Big(\underbrace{\int_0^1 \|\psi_1(\tau)\|d\tau}_{\leq \max_{t\in I}\|\psi(t)\|} + \ldots + \underbrace{\int_0^1 \|\psi_k(\tau)\|d\tau}_{\leq \max_{t\in I}\|\psi(t)\|}\Big) \leq \varrho \max_{t\in I}\|\psi(t)\| =: \varrho C$. Since $0 \in \operatorname{int}(\operatorname{Conv}(a_1,\ldots,a_k))$ by assumption, $\exists r>0: B_r(0) \subseteq \operatorname{Conv}(a_1,\ldots,a_k)$

and since $(1-\varrho)\operatorname{Conv}(a_1,\ldots,a_k) = \operatorname{Conv}((1-\varrho)a_1,\ldots,(1-\varrho)a_k)$, we also get $B_{r(1-\varrho)}(0) \subseteq \operatorname{Conv}((1-\varrho)a_1,\ldots,(1-\varrho)a_k)$. For $\varrho>0$

Here we adjust ϱ

small enough, we have $\|d\| < r/2 \Rightarrow -d \in B_{r/2}(0) \subseteq B_{r(1-\varrho)}(0) \subseteq \operatorname{Conv}((1-\varrho)a_1,\ldots,(1-\varrho)a_k) \Rightarrow -d = \tilde\alpha_1(1-\varrho)a_1 + \ldots + \tilde\alpha(1-\varrho)a_k$ for $\tilde\alpha_i \in [0,1]$

Here we adjust $\tilde\alpha_i$

with $\sum_{i=1}^{k}\tilde\alpha_i = 1$. We assign the new weights $\tilde\alpha_1(1-\varrho),\ldots,\tilde\alpha_k(1-\varrho)$ to the constant paths a_i, then we get
$$\int_0^1 \psi(\sigma)d\sigma = d + \int_{\varrho/2k}^{\varrho/2k+(1-\varrho)\tilde\alpha_1} a_1\,d\sigma + \int_{2\varrho/2k+(1-\varrho)\tilde\alpha_1}^{2\varrho/2k+(1-\varrho)\tilde\alpha_1+(1-\varrho)\tilde\alpha_2} a_2\,d\sigma + \ldots + \int_{(2k-1)\varrho/2k+(1-\varrho)\sum_{i=1}^{k-1}\tilde\alpha_i}^{(2k-1)\varrho/2k+(1-\varrho)} a_k\,d\sigma = d + \tilde\alpha_1(1-\varrho)a_1 + \tilde\alpha_2(1-\varrho)a_2 + \ldots + \tilde\alpha_k(1-\varrho)a_k = d - d = 0$$
i.e. ψ is a balanced path.

again with smooth interpolation

• Now we apply the same balancing construction to the path $\tilde\psi = \psi_1 \cdot a_1 \cdot \psi_1^{-1} \cdot \ldots \cdot \psi_k \cdot a_k \cdot \psi_k^{-1} \cdot \varphi$, where the weights of the constant paths a_i are $(1-\varrho)\alpha_i$ and the weights of all other paths are $\varrho/(2k+1)$. $\rightsquigarrow$ Set $\int_0^1 \tilde\psi(\sigma)d\sigma = 0$.

• Now we construct a genuine solution f_1 of R and the desired homotopy:

• Let $\varphi_1 := \overbrace{\psi \cdot \psi \cdot \ldots \cdot \psi}^{N-1} \cdot \tilde\psi$ be the uniform product of N factors, and $f_1 := \int \varphi_1(\sigma)d\sigma$.

• Then Property 2 implies $\|f_1\|_{C^0} = \overset{\text{will yield } C^0\text{-closedness to } 0}{\tfrac{1}{N}} \max\{\|\int\psi(\sigma)d\sigma\|_{C^0}, \|\int\tilde\psi(\sigma)d\sigma\|_{C^0}\}$ and $(f_1, \overset{\dot f_1}{\overset{\shortparallel}{\varphi_1}})$ is a genuine solution of $R = I \times \mathbb{D}^q_\varepsilon \times \Psi$

b.c. φ_1 lies completely in Ψ, since $\{\varphi, \psi_1, \ldots, \psi_k\}$ parametrize Ψ and f_1 lies in $\mathbb{D}^q_\varepsilon$ for N large enough. Moreover we have

$f_1(0) = 0 \overset{\text{Property 1}}{=} f_1(1)$ and $\varphi_1(0) = \psi(0) = \psi_1(0) = \varphi(0)$ and $\varphi_1(1) = \tilde\psi(1) = \varphi(1)$.

• For the homotopy, we utilize the homotopy $\psi_i \cdot a_i \cdot \psi_i^{-1} \sim \varphi(0)$ and deform the ψ to the constant path to $\varphi(0)$ and $\tilde\psi$ to φ.

• Utilizing the homotopy $\varphi(0)\cdot\varphi(0) \sim \varphi(0)$ and $\varphi(0)\cdot\varphi \sim \varphi$, we get a homotopy $H^1: I \times I \longrightarrow \Psi$ from φ to φ_1 s.t. $H^1(0,\tau) = \varphi(0)$

and $H^1(1,\tau) = \varphi(1)$ $\forall \tau \in I$. With this, define $H: I \times I \longrightarrow R$, $(t,\tau) \mapsto (t, \tau f_1(t), H^1(t,\tau))$, which satisfies the desired properties

$\square$

Proof of Lemma 1 for $f \equiv 0$

• Recalling Lemma 3, we can subdivide the interval I into $\overset{\exists\delta>0 \text{ uniformly } \forall t_0\in[0,1-\delta], \text{ so take } N>>0:\ 1/N<\delta \text{ and do it for } 1/N}{\text{finitely}}$ many subintervalls $[0,\tfrac{1}{N}], [\tfrac{1}{N}, \tfrac{2}{N}], \ldots, [1-1/N, 1]$

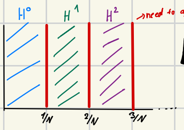
and we get that $\exists \Psi^i \subseteq \mathbb{R}^q$ flowers s.t.

i) $0 \in \operatorname{int}(\operatorname{Conv}(\partial\Psi^i))$

ii) $\psi_0^i(t) = \varphi(\tfrac{i}{N} + \tfrac{t}{N})$, $t \in I$

iii) $[\tfrac{i}{N}, \tfrac{i+1}{N}] \times \mathbb{D}^q_\varepsilon \times \Psi^i \subset R$

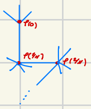
· On those intervals, we get homotopies $H^i : [\frac{i}{N}, \frac{i+1}{N}] \times I \to [\frac{i}{N}, \frac{i+N}{N}] \times \mathbb{D}^q_\varepsilon \times \Psi^i \subset R$ with desired properties. Now we glue the



homotopies together to get $H : I \times I \longrightarrow R$, which is continuous b.c. we have $H^i(\frac{i+1}{N}, \tau) \underset{H(1,\tau)=F(1)}{=} (0, f(\frac{i+1}{N})) \underset{H(0,\tau)=F(0)}{=} H^{i+1}(\frac{i+1}{N}, \tau)$.

· Moreover the maps f_τ which we get from the homotopies are still smooth b.c. differentiating f_τ on $(\frac{i}{N}, \frac{i+1}{N})$, we get

the map τf_1^i from prove of Lemma 4. This map is smooth in $[\frac{i}{N}, \frac{i+1}{N}]$ and $f_1^i(1) \underset{\text{Stem is } f(\frac{i}{N} + \frac{1}{N} t)}{=} f(\frac{i}{N} + \frac{1}{N}) \underset{t_0 = \frac{i+1}{N}}{=} f_1^{i+1}(0)$



which yields the desired regularity. For the map f_τ, we get smoothness by our construction of Ψ, $\tilde{\Psi}$ and f.

$\square$

Corollary 1: (25.3.2)

$\mathrm{Conv}_y \Omega = t \times y \times \mathbb{R}^q$

- Let $R \subseteq J^1(\mathbb{R}, \mathbb{R}^q)$ be an open ample differential relation. Then any formal solution $F = (f, \varphi): I \longrightarrow R$ which is genuine near ∂I there exists a homotopy of formal solutions, fixed near ∂I, $H: I \times I \longrightarrow R$, $(\tau, t) \mapsto F_\tau(t) = (f_\tau(t), \varphi_\tau(t))$, $F_0 = F$ such that F_1 is a genuine solution of R and f_1 is (arb.) C^0-close to f.

Proof:

- By ampleness of of R, we get that any formal solution is short, so we can apply lemma 1.

$\square$

Corollary 2: (25.3.3)

- Let $R \subseteq J^1(\mathbb{R}, \mathbb{R}^q)$ be an open and fiberwise path-connected differential relation R.

i) $\{g: I \longrightarrow \mathbb{R}^q \text{ smooth} \mid g \text{ is a genuine Sol. of } R\} \overset{C^0\text{-dense}}{\subseteq} \{f: I \longrightarrow \mathbb{R}^q \text{ smooth} \mid f \text{ is a genuine Sol. of } \mathrm{Conv}(R)\}$

ii) If additionally, R is fiberwise non-empty and ample, then $\{g: I \longrightarrow \mathbb{R}^q \text{ smooth} \mid g \text{ is a genuine Sol. of } R\} \overset{C^0\text{-dense}}{\subseteq} \{f: I \longrightarrow \mathbb{R}^q \text{ smooth}\}$.

Proof:

i) For this, we want to construct a smooth φ s.t. $\varphi(t) \in \Omega(t, f(t))$ $\forall t \in I$.

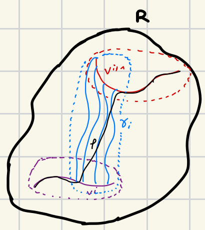
For $\tilde{t} \in I$ arb., we have that $\exists N \in \mathbb{N}$ and $v_1(\tilde{t}), \dots, v_N(\tilde{t}) \in \Omega(\tilde{t}, f(\tilde{t}))$: $f'(\tilde{t}) = \sum_{i=1}^{N} \lambda_i v_i(\tilde{t})$ b.c. f is a gen. Sol. of $\mathrm{Conv}(R)$.

- By openness of R and by cont. of f, we can extend v_1 in a small nbhd. $\mathring{I}_{\tilde{t}} \ni \tilde{t}$ to a smooth map s.t. $v_1^{\tilde{t}}(t) \in \Omega(t, f(t))$ $\forall t \in \mathring{I}_{\tilde{t}}$.

- By compactness of I, we only need finitely many $\mathring{I}_i = \mathring{I}_{t_i}$, $i = 1, \dots, N$ to cover I. To globalize our construction, we use the fiberwise path-connectedness. For $\tilde{t} \in \mathring{I}_i \cap \mathring{I}_{i+1}$, we get by path-connectedness and openness a smooth path $\gamma_i: I \longrightarrow \Omega(\tilde{t}, f(\tilde{t}))$ with $\gamma_i(0) = v^i(\tilde{t})$ and $\gamma_i(1) = v^{i+1}(\tilde{t})$. By openness R and cont. of f, we can reparametrize and extend γ_i in a small nbhd. $[\tilde{t} - \delta, \tilde{t} + \delta]$ to a smooth map $\gamma_i(\cdot)$, s.t. $\gamma_i^t(t) \in \Omega(t, f(t))$ $\forall t \in [\tilde{t} - \delta, \tilde{t} + \delta]$ and such that $\gamma_i^t(t) = v_1^i(t)$ in a nbhd. of $\tilde{t} - \delta$ and $\gamma_i^t(t) = v_1^{i+1}(t)$ in a nbhd. of $t + \delta$.

- With this, we get a smooth map φ on I s.t. $F = (f, \varphi): I \longrightarrow R$ is a short formal solution.

Applying lemma 1, we get a genuine solution g of R, which is C^0-close to f.



- For ii), note that if $f: I \longrightarrow \mathbb{R}^q$ is smooth, $\Omega(t, f(t)) \neq \emptyset$ $\forall t \in I$ by assumption and $\mathrm{Conv}(\Omega(t, f(t))) = \mathbb{R}^q$ $\forall t \in I$, we get that $\{f: I \longrightarrow \mathbb{R}^q \text{ smooth}\} = \{f: I \longrightarrow \mathbb{R}^q \text{ smooth} \mid f \text{ is a genuine Sol. of } \mathrm{Conv}\, R\}$, so the statement follows by i).

$\square$

Parametric case

• In this last part, we will state and prove the main lemma of parametric one-dimensional convex integration. The idea of the prove is the same as in the non-parametric case, so I will omit details.

Lemma 5: "Parametric one-dimensional convex integration" (25.5.1)

• Let $R \subseteq I^e \times J^1(\mathbb{R},\mathbb{R}^q)$ be an open fibered differential relation i.e. an open subset of $I^e \times \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^q$.

and $F: I^e \times I \xrightarrow{\text{section}} R$, $(p,t) \longmapsto (p, t, f(p,t), \varphi(p,t))$ be a fiberwise short formal solution of R i.e. $\forall p \in I^e$ the section

$F(p,\cdot): I \longrightarrow R_p := R \cap p \times J^1(\mathbb{R},\mathbb{R}^q)$ is a short formal solution of R_p. Suppose that $F(p,t)$ depends smoothly on $p \in I^e$ and

is a genuine solution of R_p when $p \in \mathcal{O}p(\partial I^e)$.

• Then there exists a homotopy of fiberwise short formal solutions $H: I \times I^e \times I \longrightarrow R$, $(\tau, p, t) \longmapsto F_\tau(p,t) = (f_\tau(p,t), \varphi_\tau(p,t))$

s.t. $F_0 = F$, F_1 is a genuine solution of R and s.t. $\forall \tau \in [0,1]$:

a) f_τ is (arbitrarily) $\mathcal{C}^0$-close to f

b) $F_\tau(p,0) = F(p,0)$ and $F_\tau(p,1) = F(p,1)$ $\forall p \in I^e$

c) F_τ is constantly equal to $F_0 = F$ for $p \in \mathcal{O}p(\partial I^e)$ and

d) the first derivatives of $f_1(p,t)$ w.r.t. $p \in I^e$ are (arbitrarily) $\mathcal{C}^0$-close to the respective derivatives of $f(p,t)$.

↳ Again, if F is already genuine near ∂I, then F_τ can be chosen fixed near ∂I.

• As in the non-parametric case, it is enough to consider $f \equiv 0$.

Def^n 4: Fibered flowers

• A fibered flower is a map $\psi: I^e \times S \xrightarrow{\text{abstract flower}} I^e \times \mathbb{R}^q$, as well as the set $\Psi := \psi(I^e \times S) \subseteq I^e \times \mathbb{R}^q$ parametrized by this map.

• Given a fibered flower Ψ, we will denote by Ψ_p the flower $\psi(p \times S) \subseteq p \times \mathbb{R}^q$, $p \in I^e$.

Lemma 6: "Localization of the parametric problem" (25.6.2)

• Let $R \subseteq I^e \times J^1(\mathbb{R},\mathbb{R}^q)$ be an open fibered differential relation and $F = (0,\varphi): I^e \times I \xrightarrow{\text{smooth dependence on } p} R$ be a fiberwise short formal solution of R.

Then $\exists\, \delta > 0$ s.t. $\forall t_0 \in [0, 1-\delta]$ one can choose a fibered flower $\Psi = \psi(I^e \times S) \subseteq I^e \times P_{t_0,0} = I^e \times \{t_0\} \times \{0\} \times \mathbb{R}^q$ s.t. $\forall p \in I^e$:

a) $0 \in \operatorname{int}(\operatorname{Conv}(\partial \psi_p))$

b) $\psi_0(p,t) = \varphi(p, t_0 + \delta t)$, $t \in I$

c) $p \times [t_0, t_0+\delta] \times \mathbb{D}^q_\varepsilon \times \psi_p \subseteq R_p$ for $\varepsilon > 0$ small enough

Proof:

- Let $t_0 \in [0, 1-\delta]$. For fixed $p_0 \in I^e$, $(0, f_{p_0})$ is a short formal sol. of R_p. Recall that we constructed $a_1(p_0), \dots, a_n(p_0) \in \mathrm{Conn}_{F(p_0,t_0)} R$ s.t. $0 \in \mathrm{int}(\mathrm{Conv}(a_1(p_0), \dots, a_n(p_0)))$. By openness of R, there is a small open nbhd. $\mathcal{U}$ of $p_0 \in I^e$ s.t. $a_i(p) \in \mathrm{Conn}_{F(p,t_0)} R$ is smooth and $0 \in \mathrm{int}(\mathrm{Conv}(a_1(p), \dots, a_n(p)))$ $\forall p \in \mathcal{U}$. Now again by openness of R, we may define smooth paths $\psi_i(p,t)$ in $\mathrm{Conn}_{F(p,t_0)} R$ from $f(p,t_0)$ to $a_i(p)$ s.t. $\psi^{\mathcal{U}} = (f(p, t_0 + \delta t), \psi_1(p,t), \dots, \psi_n(p,t))$ is a flower fibered over $\mathcal{U}$. (will be chosen later. Imagine it to be 0 for now if that makes you uneasy)
- By compactness of I^e we can choose a finite covering of I^e by open sets $\mathcal{U}_j$, $j = 1, \dots, L$, such that we have above flower $\psi^{\mathcal{U}_j} = (f(p, t_0 + \delta t), \psi_1^{\mathcal{U}_j}(p,t), \dots, \psi_{n_j}^{\mathcal{U}_j}(p,t))$ fibered over $\mathcal{U}_j$. Let $\mathcal{U}_j' \subset\subset \mathcal{U}_j$ (compactly contained i.e. $\overline{\mathcal{U}_j'} \subseteq \mathcal{U}_j$) for $j = 1, \dots, L$ s.t. $I^e \subseteq \bigcup_{j=1}^{L} \mathcal{U}_j'$ be slightly smaller open sets. For every $j = 1, \dots, L$ choose a smooth cut-off function $\beta^j : I^e \longrightarrow [0,1]$ s.t. $\beta^j \equiv 1$ on $\mathcal{U}_j'$ and $\beta^j \equiv 0$ on $I^e \setminus \mathcal{U}_j$
- With those, we define $\psi_i^j : I^e \times I \longrightarrow I^e \times P_{t_0,0}$, $(p,t) \longmapsto \begin{cases} \psi_i^{\mathcal{U}_j}(p, \beta^j(p) t) & \text{for } p \in \mathcal{U}_j \\ f(p,t_0) & \text{for } p \in I^e \setminus \mathcal{U}_j \end{cases}$ for $i = 1, \dots, N_j$, $j = 1, \dots, L$

and $\Psi := (f(p, t_0 + \delta t), \psi_1^1, \psi_2^1, \dots, \psi_{N_1}^1, \psi_1^2, \psi_2^2, \dots, \psi_{N_2}^2, \dots, \psi_1^L, \dots, \psi_{N_L}^L)$, where we can choose $\delta > 0$ and $\varepsilon > 0$ by openness of R and compactness of I.

$\square$

- With above Lemma 6, we get that it is enough to prove the parametric one-dimensional convex integration for the case when:

i) the fibered relation R consists of fibers $R_p = p \times I \times \mathbb{D}^q_\varepsilon \times \Psi_p \subseteq p \times J^1(\mathbb{R}, \mathbb{R}^q)$, $p \in I^e$ where $\Psi = \psi(S \times I^e) \subseteq I^e \times \mathbb{R}^q$ is a fibered flower s.t. $0 \in \mathrm{int}(\mathrm{Conv}(\partial \psi_p))$ for each $p \in I^e$

ii) $F = (0, f) : I^e \times I \longrightarrow R$ where f is the stem of Ψ. (25.6.1)

- In order to prove above reduction we do the following 3 steps:

Convex decomposition of a section

- Let Ψ be a fibered flower and let $a_i(p) = \psi_i(p,1)$, $i = 1, \dots, N$ and let $\Delta_p := \mathrm{Conv}\, \partial \psi_p$ for $p \in I^e$.
- Let $d : I^e \longrightarrow I^e \times \mathbb{R}^q$, $p \longmapsto (p, d(p)) \in p \times \mathrm{int}(\Delta_p)$ be a map. Then there exist smooth maps $\alpha_i : I^e \longrightarrow [0,1]$, $i = 1, \dots, N$ s.t.

$\alpha_1(p) + \dots + \alpha_N(p) = 1$ and $\alpha_1(p) a_1(p) + \dots + \alpha_N(p) a_N(p) = d(p)$ (construct them locally (we are in an open set) and globalize them by part. of unity)

Construction of the homotopy F_τ

- We will use the proof for the non-parametric case (see proof of lemma 1):
- Take weights $(1-\rho)\alpha_i(p)$ (smooth) and $\rho/2K$ to define $\psi = \psi_1 \cdot a_1 \cdot \psi_1^{-1} \cdot \ldots \cdot \psi_N \cdot a_N \cdot \psi_N^{-1}$ (from the flower) as befor (p dependence) $\rightsquigarrow$ Get smooth $d(p)$ with $\|d(p)\| \le C\rho$.

- Use that $r(p) := \sup\{r>0 \mid B_r(0) \subseteq \Delta_p\}$ is cont. on I^e to get uniform $\delta > 0$ s.t. $d(p) \in \operatorname{int}((1-\delta)\Delta_p)$, use convex decomp

to get new $\tilde{\alpha}_i(p)$ to balance $\psi(p,t)$ $\forall p \in I^e$. Do the same for $\tilde{\psi} = \psi_1 \cdot a_1 \cdot \psi_1^{-1} \cdot \ldots \cdot \psi_N \cdot a_N \cdot \psi_N^{-1} \cdot f$.

- Again, let $\varphi_1 = \psi \cdot \ldots \cdot \psi \cdot \tilde{\psi}$ uniform product of N factors and $f_1(p,t) = \int_0^t \varphi_1(p,\sigma)\,d\sigma$. Define homotopies as befor

and get desired homotopy F_τ, which satisfies property a) and b).

- For property c), let $\beta : I^e \longrightarrow [0,1]$ s.t. $\beta \equiv 0$ near ∂I^e and $\beta \equiv 1$ on smaller cube $\subseteq I^e$.

- Then we redefine the homotopy $H : I \times I^e \times I \longrightarrow \mathbb{R}$, $(\tau, p, t) \longmapsto F_{\beta(p)\tau}(p,t)$.

Derivatives with respect to the parameter

- In order to show property d), we need to show that $\partial_p f_1(p,t)$ arb. $\mathcal{C}^0$-close to 0.

- We have $\partial_p f_1(p,t) = \partial_p \int_0^t \varphi_1(p,\sigma)\,d\sigma \underset{\partial_p \varphi_1 \text{ is bdd.}}{=} \int_0^t \partial_p \varphi_1(p,\sigma)\,d\sigma$ and $\partial_p \varphi_1(p,t) \overset{\text{fix } t \text{ and convince yourself}}{=} \partial_p \psi(p,t) \cdot \ldots \cdot \partial_p \psi(p,t) \cdot \partial_p \tilde{\psi}(p,t)$

- Since ψ and $\tilde{\psi}$ are $\overset{\forall p \in I^e}{\text{balanced}}$ $\Rightarrow$ $\partial_p \psi$ and $\partial_p \tilde{\psi}$ are balanced b.c. $\int_0^1 \partial_p \psi(p,\sigma)\,d\sigma = \partial_p \int_0^1 \psi(p,\sigma)\,d\sigma \overset{\psi \text{ balanced}}{=} 0$.

- Using property 2, we get that for any $t \in I$, $p \in I^e$:

$$|\partial_p f_1(p,t)| \le \frac{1}{N} \max\left\{ \sup_{t \in I} |\partial_p g(p,t)|, \sup_{t \in I} |\partial_p h(p,t)| \right\} \le \frac{1}{N} \max\left\{ \|\partial_p g\|_{\mathcal{C}^0}, \|\partial_p h\|_{\mathcal{C}^0} \right\}$$

$\Rightarrow$ $\|\partial_p f_1\|_{\mathcal{C}^0} \le \frac{1}{N} \max\{\|\partial_p g\|_{\mathcal{C}^0}, \|\partial_p h\|_{\mathcal{C}^0}\}$, where $g(p,t) = \int_0^t \psi(p,\sigma)\,d\sigma$, $h(p,t) = \int_0^t \tilde{\psi}(p,\sigma)\,d\sigma$, So we get the desired statement.

$\square$