# Homotopy Principle for Ample Differential Relations

<u>last time</u>: • ample set $\Omega \subset P$ ~ affine space, if $\forall y \in \Omega$: $\mathrm{Conv}_y \Omega$ = convex hull of path conn. comp. of $y \in \Omega = P$

• ample diff. rel: $R \subseteq J^1(\mathbb{R},\mathbb{R}^q) \cong \mathbb{R}\times\mathbb{R}^q\times\mathbb{R}^q$ if $\forall (t,y)\in \mathbb{R}\times\mathbb{R}^q$ : $\Omega(t,y) := R \cap \underbrace{\left((t,y)\times\mathbb{R}^q\right)}_{=p_1^{-1}((t,y))\cong \mathbb{R}^q}$ is <u>ample</u>

$J^1(\mathbb{R},\mathbb{R}^q) \xrightarrow{p_1} J^0(\mathbb{R},\mathbb{R}^q) = \mathbb{R}\times\mathbb{R}^q$

• COR: one dimensional convex integration:

Let $R \subseteq J^1(\mathbb{R},\mathbb{R}^q)$ <u>ample diff. rel.</u> $\Rightarrow$ $\forall F_0 = (f,\varphi): I \to R$ formal sol. which is genuine near $\partial I$

$\Rightarrow \exists$ formal sols $F_\tau = (f_\tau, \varphi_\tau): I \to R$, $\tau\in[0,1]$ w/ $F_1$ genuine sol & $f_1$ $\mathcal{C}^0$-close to $f$

<u>today</u>: want similar result but w/ $J^1(\mathbb{R}^n,\mathbb{R}^q)$ or $X^{(1)}$ instead $J^1(\mathbb{R},\mathbb{R}^q)$

## 1. Case: $J^1(\mathbb{R}^n,\mathbb{R}^q)$ – Iterated convex Integration:

need: new def. of ample diff. rel: "coordinatewise"

$\Rightarrow$ DEF: $z = (v,w,z_1,\dots,z_n) \in \mathbb{R}^n\times\mathbb{R}^q\times\underbrace{\mathbb{R}^q\times\dots\times\mathbb{R}^q}_{(n-1)} = J^1(\mathbb{R}^n,\mathbb{R}^q)$ then the <u>i'th coordinate principal subspace</u> at $z$ is:

$P^i(z) =: (v,w,z_1,\dots,z_{i-1})\times\mathbb{R}^q\times(z_{i+1},\dots,z_n)$ $\quad i = 1,\dots,n$ ~ fix everything but the i'th partial derivative!

RMK: for $n=1$ we had: $z = (v,w,z_1)$ $\Rightarrow$ only one principal subspace: $P^1(z) = (v,w)\times\mathbb{R}^q$

DEF: diff. rel. $R \subset J^1(\mathbb{R}^n,\mathbb{R}^q)$ is <u>ample in coordinate direction</u> if $\forall k = 1,\dots,n$ $\forall z\in J^1(\mathbb{R}^n,\mathbb{R}^q)$

$\underbrace{P^i(z)\cap R}_{\text{all pnts in } R \text{, which have the same coords as } z \text{ except for the i'th partial derivative!}} \subset P^i(z)$ is ample

RMK: it is enough to take $z\in R$ for the ample in coord. dir. definition, since if $z\in J^1(\mathbb{R}^n,\mathbb{R}^q)\setminus R$ then either $P^i(z)\cap R = \emptyset$ or $\exists$ $z'\in P^i(z)\cap R$ (especially $z'\in R$) & then $P^i(z') = P^i(z)$ i.e. $P^i(z')\cap R = P^i(z)\cap R$

EXP: 1) for $n<q$: $R_{imm} = \{(\overbrace{v,w}^{J^0(\mathbb{R}^n,\mathbb{R}^q)}, \overbrace{A}^{\mathbb{R}^{q\times n}}) \in J^1(\mathbb{R}^n,\mathbb{R}^q) \mid v\in\mathbb{R}^n, w\in\mathbb{R}^q, \mathrm{rk}(A) = n\} \subset J^1(\mathbb{R}^n,\mathbb{R}^q)$

For $z = (v,w,A)\in R_{imm}$: $\underbrace{P^i(z)\cap R}_{=\{(v,w,B)\in J^1(\mathbb{R}^n,\mathbb{R}^q) \mid \text{the i'th column of } B \text{ is arbitrary, all the others equal to the ones of } A\} \cong \mathbb{R}^q} = P^i(z)/L$ w/ $L = \langle$columns of $A$, except i'th column$\rangle \subseteq P^i(z)$ ~ choice of the i'th column

$\Rightarrow$ since $\dim L = n-1$ ($n-1$ - indep. vectors, since $z\in R_{imm}$) $\Rightarrow$ $\mathrm{codim}(L)\geq 2$; i.e. $P^i(z)/L$ path. conn!

& $\forall l \in L$ can be written as a convex combination of two vectors not in $L$

$\Rightarrow \mathrm{Conv}(P^i(z) \cap R) = P^i(z)$

2) $n \geq q$: $R_{subm} = \{(v, q, A) \in J^0(\mathbb{R}^n, \mathbb{R}^q) \times M_{q \times n} \mid \mathrm{rank} A = q\}$

For $z = (x, y, A) \in R_{subm}$ & supp. $i \in \{1, \dots, n\}$ is s.th. i'th column of $A$ is lin. ind. of the other columns

$\Rightarrow P^i(z) \cap R = P^i(z)/L$ w/ $L = \langle$columns of $A$, except the i'th column$\rangle$ & $\mathrm{codim}\, L = 1$

$\Rightarrow P^i(z)$ gets seperated into two $\Rightarrow$ not ample

THM: Convex integration over a cube:

$R \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ open diff. rel. ample in coordinate directions &

$F_0 = (f_0, \varphi_0): I^n \to R \subset J^0(\mathbb{R}^n, \mathbb{R}^q) \times M_{q \times n}$ formal solution of $R$ which is genuine on $Op\, \partial I^n$

$\Rightarrow \exists$ hmtpy of formal sols $F_\tau = (f_\tau, \varphi_\tau): I^n \to R$, $\tau \in [0,1]$ w/ $F_1$ genuine sol of $R$, s.th. $\forall \tau$:

a) $f_\tau$ is $C^0$-close to $f$

b) $F_\tau|_{Op\, \partial I_n} = F|_{Op\, \partial I_n}$

$\Rightarrow$ COR: h-principle for ample differential relations over a cube

Let $R \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ open diff. rel. over $I^n$ ample in the coord. directions

$\Rightarrow$ relative h-principle hold for $R$ over $(I^n, \partial I^n)$ & $(I^n, A)$ for $A \subset \partial I^n$ closed

Pf works analoge to the one of the thm

proof of THM: $\varphi = (\varphi^1, \dots, \varphi^n)$. Want to integrate $F = (f, \varphi)$ coordinate-wise, using Lem 25.5.1.

**25.5.1. (Parametric one-dimensional convex integration)** *Let $\mathcal{R} \subset I^l \times J^1(\mathbb{R}, \mathbb{R}^q)$ be an open fibered differential relation (see Section 7.2.E) and*

$$F = F(p,t) = (f(p,t), \varphi(p,t)) : I^l \times I \to \mathcal{R}$$

*be a fiberwise short formal solution of $\mathcal{R}$, i.e., for each $p \in I^l$ the section*

$$F(p,t) : p \times I \to \mathcal{R}_p = \mathcal{R} \cap p \times J^1(\mathbb{R}, \mathbb{R}^q)$$

*is a short formal solution of $\mathcal{R}_p$. Suppose that $f(p,t)$* **smoothly** *depends on $p$ and consists of genuine solutions of $\mathcal{R}_p$ when $p \in \mathcal{O}p\, \partial I^l$. Then there exists a homotopy of fiberwise short formal solutions*

$$F_\tau = F_\tau(p,t) = (f_\tau(p,t), \varphi_\tau(p,t)) : I^l \times I \to \mathcal{R},\ \tau \in [0,1],$$

*which joins $F_0 = F$ with a genuine solution $F_1$ of $\mathcal{R}$ such that for all $\tau$*

(a) *$f_\tau$ is (arbitrarily) $C^0$-close to $f$;*
(b) *$F_\tau(p,0) = F(p,0)$ and $F_\tau(p,1) = F(p,1)$ for all $p \in I^l$;*
(c) *$F_\tau$ is constant for $p \in \mathcal{O}p\,(\partial I^l)$, and*
(d) *the first derivatives of $f_1(p,t)$ with respect to the parameter $p$ are (arbitrarily) $C^0$-close to the respective derivatives of $f(p,t)$.*

1.) regard $I^n$ as $\{I \times p$ parallel to $x_1$-axis $\mid p \in I^{n-1}\}$ & def. fibered diff. rel. $R^1 \subset I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q)$ s.th.

for $t = x_1$ , $p = (x_2, \dots, x_n)$:

$$\Omega_p(f(t,p)) := R^1_p \cap (p \times (t, f(t,p)) \times \mathbb{R}^q) = p \times (t, f(t,p)) \times \underbrace{\mathrm{Conn}_{F(t,p)}(\overbrace{R \cap P^1(F(t,p))}^{\subseteq \mathbb{R}^q})}_{\subseteq \mathbb{R}^q} \subseteq J^1(\mathbb{R}, \mathbb{R}^q)$$

path-conn. comp.

•) want: $R_p$ open $\Rightarrow$ expand $R_p$ to a small nbhd of graph$(f)$

i.e. define $\Omega_p((t,y)) := R^1_p \cap (p \times (t,y) \times \mathbb{R}^q)$ for $y$ close to $f(t,p)$:

Decrease $\Omega_p(f(t,p))$ s.th. still ample & take:

$$D^q_\varepsilon(f(x)) \times \Omega_p(f(x)) \subset R \subset J^1(\mathbb{R}, \mathbb{R}^q) \quad \text{for suff. small } \varepsilon.$$

I don't quite understand this definition here

2. Apply 25.5.1. to $(f(t,p), \varphi^1)$ which is short since $R$ is ample in coord. directions

Get genuine solution $(f^1(t,p), \partial_t f^1(t,p))$ of $R^1$ and hence formal sol.

$$F^1 = (f^1; \partial_{x_1} f^1, \varphi^2, \dots, \varphi^n)$$

of the relation $R$. We have

•) $F^1$ homotopic to $F$ in $R$

•) by a) $f^1$ is $C^0$-close to $f$

•) by b) & c) $f^1$ coincides w/ $f$ near $\partial I^n$

•) $F^1$ holonomic w.r.t. coordinate $x_1$

3. regard $I^n$ as family of intervals parallel to $x_2$ $\rightsquigarrow$ form relation $R^2$ fibered over $I^{n-1}$

$\Rightarrow$ get formal sol. $F^2 = (f^2; \partial_{x_1} f^1, \partial_{x_2} f^2, \varphi^3, \dots, \varphi^n)$ of $R$ as above

& by d) the section $\partial_{x_1} f^2$ is $C^0$-close to $\partial_{x_1} f^1$

$\Rightarrow$ can deform formal solution $F^2$ by linear homotopy in $R$ to formal sol:

$$\tilde{F}^2 = (f^2, \partial_x f^2, \partial_x f^2, \varphi^3, \dots, \varphi^n) \quad \text{which is holonomic in } x_1 \,\&\, x_2$$

4. repeat: $(f_0; \varphi^1, \varphi^2, \dots, \varphi^n) \longrightarrow (f^1; \partial_{x_1} f^1; \varphi^2, \dots, \varphi^n) \longrightarrow \dots \to (f^n; \partial_x f^n, \dots, \partial_{x^n} f^n)$ ☒

## 2. Case: $X^{(1)}$ – Convex integration of ample differential relations

Now: fibration $p: X \to V$ usually $X = V \times W$

Recall: •)
$$\begin{array}{lll} X^{(1)} = V \times \mathrm{Sec}(X)/\sim_1 & [(v,f)] & \text{w/ } (v,f) \sim_1 (v,g) \Leftrightarrow f(v) = g(v) \,\&\, d_v g = d_v f \\ \downarrow p^1_0 & \downarrow & \\ X = X^0 = V \times \mathrm{Sec}(X)/\sim_0 & [(v,f)] & \text{w/ } (v,f) \sim_0 (v,g) \Leftrightarrow f(v) = g(v) \in X \end{array}$$

global sections, but we only consider them locally at $v$

$f(v) \leftarrow\!\shortmid (v,f)$

$x \longmapsto (p(x), g)$ w/ $g(p(x)) = x$

•) what is $(p^1_0)^{-1}(x) =: E_x$ for $x \in X$? Let $v = p(x)$

$$E_x = \{[v,f] \in X^{(1)} \mid f(v) = x\} = \{d_v f : T_v V \to T_x X \mid f \in \mathrm{Sec}(X), f(v) = x\}$$

fixed $v \in V$, $f(v) \in X$ only thing that can vary is $d_v f$!

$$= \{\sigma : T_v V \to T_x X \mid \underline{dp_x \circ \sigma = \mathrm{id}_{T_v V}}\} = \mathrm{Hom}(T_v V, \mathrm{Vert}_x)$$

vector spaces of same dimension

$f \in \mathrm{Sec}(X) \Leftrightarrow f: V \to X$ & $p \circ f = \mathrm{id}$

on a horizontal lift $\sigma$ is fixed by $dp_x \circ \sigma = \mathrm{id}_{T_v V}$

What are the "coordinate principal subspaces" in this case?

DEF: choose hyperplane $\tau \subset T_v V$ & linear map $\ell : \tau \longrightarrow \mathrm{Vert}_x$

Define $P_\tau^\ell = \{ L \in \mathrm{Hom}(T_v V, \mathrm{Vert}_x) \mid L|_\tau = \ell \}$.

RMK: for $V = \mathbb{R}^n$, $X = \mathbb{R}^n \times \mathbb{R}^q$, for $x = (v, w) \in X$ : $T_v V = \mathbb{R}^n$, $\mathrm{Vert}_x = T_w \mathbb{R}^q \cong \mathbb{R}^q$ ; define: $\tau : \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$ $(x_1, \dots, x_{n-1}) \longmapsto (x_1, \dots, x_{n-1}, 0)$

& $\ell : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^q$ linear, i.e. $\ell = \begin{pmatrix} | & & | \\ z_1 & \cdots & z_{n-1} \\ | & & | \end{pmatrix} \in \mathbb{R}^{q \times (n-1)}$ w/ $z_i \in \mathbb{R}^q$ fixed (partial derivatives)

$\hookrightarrow$ as a matrix

$\Rightarrow P_\tau^\ell = \{ L \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q) \mid L|_\tau = \ell \} = \left\{ L = \begin{pmatrix} | & & | & | \\ z_1 & \cdots & z_{n-1} & x \\ | & & | & | \end{pmatrix} \;\middle|\; x \in \mathbb{R}^q \right\}$

DEF: $\mathcal{R} \subset X^{(1)}$ is called <u>ample</u> if $\mathcal{R}$ intersects $\forall P_\tau^\ell$ along ample sets

RMK: Ampleness in coordinate directions is less restrictive than ampleness

EXP: $\mathcal{R}_{imm}$ is ample for $n < q$. $\mathcal{R}_{subm}$ is not ample

THM (Homotopy principle for ample differential relations)

Let $\mathcal{R} \subset X^{(1)}$ be an open ample diff. rel. $\Rightarrow \forall$ forms of the h-principle hold for $\mathcal{R}$

pf: reduce to the relative h-principle by induction over skeleta of a triangulation of the base $V$