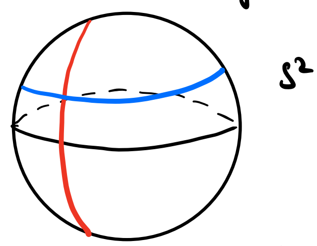
Directed immersion & embeddings

I. Directed immersions & ampleness

What is a directed immersion?

Let's start with the following: Assume, you want to embed a line-segment into S^2.

There are actually several ways to do this:



Essentially, the different embedded line-segments are going along different directions. We now want to fix the possible directions an embedding (or more generally an immersion) can go along within the manifold our embedding is living.

We are going to direct the immersion along certain subspaces! To do this, let's look at the following:

Def.: The Grassmannian $Gr_k(V)$ of a n-dim vector space V (with diff. structure) is a smooth manifold, that parametrizes the set of all k-dim linear subspaces of V.

Def.: Considering a vector bundle $\pi: E \to M$ (some mfd. M), we define the Grassmann-bundle

$$\tilde{\pi}: Gr_k(E) \longrightarrow M \quad \text{by} \quad Gr_k(E) := \coprod_{x \in M} Gr_k(E_x).$$

It earns its smooth structure per definition from the fibers being the Grassmannians of the E_x.

Furthermore, consider a smooth q-dim. mfd W with it's tangen bundle. We define the Grassmann-bundle of tangent n-planes to W $Gr_n(W)$, by taking the Grassmann-bundle of the tangent bundle:

$$TW \xrightarrow{\pi} W \rightsquigarrow Gr_n(TW) \xrightarrow{\tilde{\pi}} W$$

Def.: For some smooth n-mfd. V, q-mfd. W & a cont. map $F: TV \longrightarrow TW$ we define the <u>tangential lift</u> of F, as

$$GF: V \longrightarrow Gr_n(W).$$

Let $A \subseteq Gr_n(W)$ be some arbitrary subset. An immersion $f: V \longrightarrow W$ is called <u>A-directed</u>, if

$$G(df) \subseteq A.$$

Notation: Given $A \subset Gr_n(W)$, we'll denote by

- $\mathcal{R}_A := \{ f: V \to W \mid f \text{ is } A\text{-dir. imm.} \} \subset \mathcal{R}_{imm}$
- $A_w = A \cap Gr_n(T_w W)$ for $w \in W$
- $Gr_{n-1}(A_w) = \bigcup_{L \in A_w} Gr_{n-1}(L) \subset Gr_{n-1}(T_w W)$

recall: $\mathcal{R} \subset X^{(r)}$ is ample, iff it intersects all principle subspaces along ample sets. (i.e. the intersection with all principle subspaces is ample)

Proposition 1 (Ampleness-Criterion):

$\mathcal{R}_A$ ample iff $\forall w \in W$ $\forall S \in Gr_{n-1}(A_w)$ $\Omega_S := \{ v \in T_w W \mid \text{Span}(S, v) \in A_w \} \subset T_w W$ ample.

proof:

„⇐": first note, that every principle subspace is a coordinate principle subspace for a suitable set of charts, therefore we'll just consider $J^1(\mathbb{R}^n, \mathbb{R}^q)$.

For $p = (x, y, a) \in \mathcal{R}_A$, let $P = P^{(i)}(p)$ be coordinate principle subspace:

$$P^{(i)}(p) = (x, y, v_1, \ldots, v_{i-1}) \times \mathbb{R}^q \times (v_{i+1}, \ldots, v_n) \qquad \text{for } a = (v_1, \ldots, v_n)$$

$\rightsquigarrow P^{(i)}(p) \cap \mathcal{R}_A \cong T_y \mathbb{R}^q \cong \mathbb{R}^q$

Let $S^{(i)} = (v_1, \ldots, \hat{v}_i, \ldots, v_n) \in (T_y \mathbb{R}^q)^{n-1}$.

An immersion $f: V \to W$ is <u>A-directed</u>, if $G(df) \subset A$, i.e. locally $\forall x \in V$

$$df(T_x V) \subset A_y, \quad \text{i.e.} \quad im(d_x f) = span(S^{(i)}, v_i) \subset A_y \subset Gr_n(T_w W)$$

$(\Rightarrow f \text{ immersion})$

$\Rightarrow p^{(i)}(p) \cap R_A = \{ v \in T_w W \mid Span(S^{(i)}, v_i) \subset A_w \} = \Omega_{S^{(i)}}$

If $\Omega_{S^{(i)}}$ ample $\forall S^{(i)} \Rightarrow R_A$ ample

"$\Rightarrow$": follows from $Diff(V)$-invariance of R_A, since it is then enough to check condition locally in charts as above $\square$

II. Directed immersions into almost symplectic manifolds

recall: 1) an almost symplectic structure on mfd. W is a non-degenerate 2-form on its tangent bundle

2) we call a subspace $A \subset (T_q W, \omega)$

- symplectic, if $A \cap A^\omega = \{0\}$
- Lagrangian, if $A^\omega = A$
- isotropic, if $A \subseteq A^\omega$
- coisotropic, if $A^\omega \subseteq A$

$A^\omega = \{ w \in T_q W \mid \omega(a, w) \ \forall a \in A \}$

Def.: We define symplectic/lagrangian, isotropic/coistropic immersions trough A-directed immersions with $A = \{ Z \in Gr_n(W) \mid Z \text{ sympl.}/\ldots \}$

rem.: R_{sympl} open, $R_{lagr}, R_{iso}, R_{coiso}$ not open

R_{imm} is open, since the desired property $rk(df) = n$, i.e. $\det(df) \neq 0$ is open condition

$R_{sympl} \subset R_{imm}$: consider $Z \subset T_q W$ symplectic, i.e. $Z \cap Z^\omega = \{0\}$

$\rightsquigarrow v \in Z$ s.t. $\omega(v, w) = 0 \ \forall w \in T_q W \Rightarrow v = 0$

$\Rightarrow \omega|_{Z \setminus \{0\}} \neq 0$ open condition $\Rightarrow R_{sympl}$ open

$R_{lagr} \subset R_{imm}$: consider $Z \subset T_q W$ Lagrangian, i.e. $Z = Z^\omega$

$\Rightarrow \omega|_Z = 0$ closed cond. $\Rightarrow R_{lagr}$ closed

Def.: Assume W is equipped with a Riemann. metric.
Define $\mathcal{R}^\varepsilon_{lagr}$, $\mathcal{R}^\varepsilon_{iso}$, $\mathcal{R}^\varepsilon_{coiso}$ through A^ε-directed immersions for A^ε the open ε-nbhd. of the Lagr. / iso. / coiso. set $A \subset Gr_n(W)$.

rem.:
- $\mathcal{R}^\varepsilon_{lagr}$, $\mathcal{R}^\varepsilon_{iso}$ ample $\Rightarrow$ h-principle holds
- $\mathcal{R}_{symp}$, $\mathcal{R}^\varepsilon_{coiso}$, $\mathcal{R}^\varepsilon_{isosym}$ not ample

III. Directed immersions into almost complex manifolds

recall: 1) An almost complex structure on a manifold W is is a complex structure on the tangent bundle $J : T_pW \longrightarrow T_pW$, $J^2 = id$
$\rightsquigarrow$ turns T_pW into complex v.s.

2) $S \subseteq \mathbb{C}^n$ is called
- complex, if $iS = S$
- real, if $S \cap iS = \{0\}$
- co-real, if $S + iS = \mathbb{C}^n$

Def.: Let W be a manifold of $\dim(W) = 2k$ with almost cmplx. structure.
We define <u>complex / real / co-real immersions</u> trough A-directed immersions with $A = \{ z \subset Gr_n(W) \mid A \text{ is complex} / \text{—} \}$

Def.: Assume W is equipped with a Riemann. metric.
Define $\mathcal{R}^\varepsilon_{comp}$ through A^ε-directed immersions for A^ε the open ε-nbhd. the complex subset $A \subset Gr_n(W)$.

rem.:
- $\mathcal{R}^\varepsilon_{comp}$ is not ample

rem.: $R \subset X^{(1)}$, $R = X^{(1)} \setminus \Sigma_1$ and $\forall a \in \Sigma_1$ any principle subspace $P(a)$
$P \cap \Sigma_1 \subset P$ smooth submfd. of $\operatorname{codim}(\Sigma_1) \geq 2$, then we call
$\Sigma_1 \subset X^{(1)}$ a thin singularity

Proposition 2: $\Sigma \subset X^{(1)}$ thin singularity $\Rightarrow R = X^{(1)} \setminus \Sigma$ ample

Theorem 1 (Gromov): R_{real} & R_{coreal} are ample and hence all forms of the h-principle hold for real & co-real immersion $V \longrightarrow (W, J)$.

proof: R_{real}: Since we want to embed V with $\dim(V) = n$ into a W with $\dim(W) = 2k$ along only real subspaces, we may w.l.o.g. assume $n \leq k$ (p. 222)

For some $(n-1)$-dim subspace $S \subset L \in A_{real}$ of a n-dim real subset $L \in A_{real}$, determine $\Omega_S = \{ v \in T_w W \mid \operatorname{span}(S, v) \in A_{real} \}$:

what constraints for $v \in T_w W$, s.t. $Z := \operatorname{span}(S, v)$, $Z \cap iZ = \{0\}$?

assume $Z \cap iZ \neq \{0\} \Rightarrow \exists \overset{\in Z}{z} = s + \lambda v = is' + i\mu v = z' \in iZ$

$$\Leftrightarrow s - is' = (i\mu - \lambda) v$$

$$\Leftrightarrow v = \frac{s - is'}{(i\mu - \lambda)} \in S + iS$$

$\Rightarrow \Omega_S = T_w W \setminus (S + iS)$ with $\dim(S + iS) = 2(n-1) \leq 2 + 2k$

$\Rightarrow S + iS$ thin singularity $\Rightarrow R_{real}$ ample

$R_{co\text{-}real}$: similarly: $Z = \operatorname{span}(S, v)$, assume $Z + iZ = T_w W$

pick $w \in T_w W = Z + iZ \rightsquigarrow w = s + \mu v + is' + i\lambda v$

$$= \underbrace{s + is}_{\in S + iS} + (\mu + i v) v$$

$S + iS = T_w W$: v can be anything, except $v \in S$ $(\operatorname{rank}(df) = n) \Rightarrow \Omega_S = T_w W \setminus S$

$S + iS \neq T_w W$: choose $w \in T_w W \setminus (S + iS) \rightsquigarrow v \in T_w W \setminus (S + iS) \Rightarrow \Omega_S = T_w \setminus (S + iS)$

$\operatorname{codim}(S) \geq 2$, $\operatorname{codim}(S + iS) \geq 2 \overset{(\text{Prop. 2})}{\Longrightarrow} R_{co\text{-}real}$ ample $\square$

IV. Directed embeddings

<u>Def.:</u> $R_A \subset J^1(V,W)$ is called <u>affine ample</u>, if for any $S \in Gr_{n-1}(A_w)$, any hyperplane $H \supset S$ and any affine hyperplane $H' \subset T_wW$ parallel to H, the set $\Omega_S \cap H'$ is ample in H'

<u>rem.:</u> If a diff. rel. is the complement of a thin singularity, then it is affine ample.

Using this and the proof of thm. 1, we have that

R_{real}, $R_{co\text{-}real}$ are ample

<u>Theorem 2 (Directed embeddings):</u>

Suppose $A \subset Gr_n(W)$ is an open subset and the corresponding $R_A \subset J^1(V,W)$ is affine ample. Then every embedding $f_0 : V \longrightarrow W$ whose tangential lift

$$G_0 = G(df_0) : V \longrightarrow Gr_n(W)$$

is homotopic over V to a map $G_1 : V \longrightarrow A$ can be isotoped to an A-directed embedding $f_1 : V \longrightarrow W$. Moreover, $f_t : V \longrightarrow W$ can be chosen arbitrarly C^0-close to the constant isotopy.

<u>rem.:</u> Let's consider $G_0 = G(df_0) : v \longmapsto Z \subset T_{f_0(v)}W$.

„Homotopic over V", means, that for $G_t : V \longrightarrow Gr_n(W)$, the underlying homotopy g_t is chosen constant, i.e. $g_t \equiv f_0$.

In general, one could look at G_t homotopic over embeddings, where g_t is not fixed. The isotopy f_t can then be chosen arbitrarly C^0-close to g_t. We restrict to $g_t = f_0$.

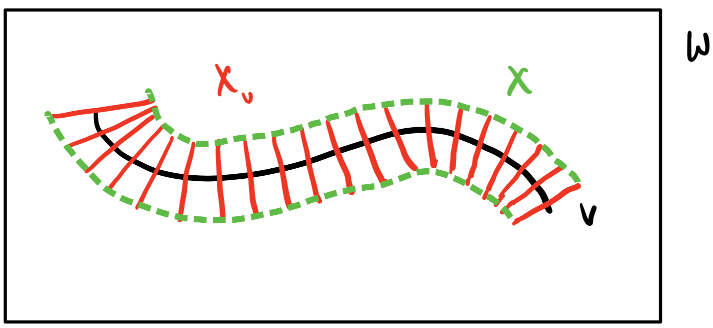
proof of thm. 2

To prove theorem 2, we are going to use two auxillary Lemmas. The first one in particular is going to give us some intuition on what affine ampleness is and how its power comes into play.

Def.: Given smooth manifolds $V \subseteq W$, we define a tubular neighborhood X of V in W (fibered over V) as a vector bundle

$$\pi : X \longrightarrow V \quad \text{together with a map} \quad I : X \longrightarrow W, \ \text{s.t.}$$

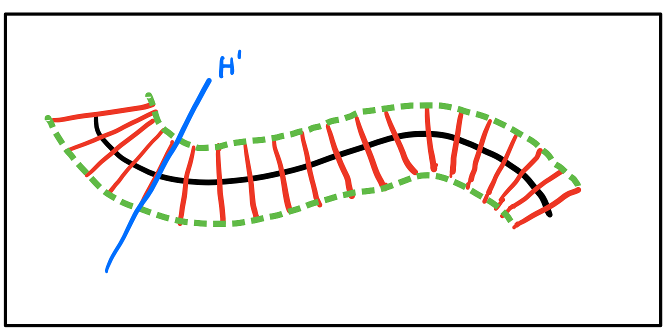
- $I \circ 0_X = i$ with $i : V \hookrightarrow W$ embedding, 0_X zero-section
- $\exists\, U \subseteq X$, $\tilde{U} \subseteq M$ with $0_X(V) \subseteq U$, $V \subseteq \tilde{U}$ s.t. $I|_U : U \longrightarrow \tilde{U}$ diffeo.



Lemma 1: Let $A \subset Gr_n(W)$ be an open subset, s.t. the corresponding $R_A \subset J^1(V,W)$ is affine ample. Let $V \subset W$ be an embedded mfd. and X a tubular nbhd. of V, fibered over V.

Then the relation $R_A^X \subset X^{(1)}$ of A-directed sections of the fibration $p : X \longrightarrow V$ is open and ample.

sketch of proof:



We slice with these affine hyperplane H' (in coordinate direction for fibers X_v), to get that the resulting $\Omega_s \cap H'$ is ample for every such H'.
Then the diff. rel. w.r.t. X is ample. □

Now we use Lemma 1, to prove the following statement:

<u>Lemma 2</u>: Let $A \subset Gr_n(W)$ be open, s.t. $R_A \subset J^1(V,W)$ affine ample.
Let $\phi^t : TW \longrightarrow TW$ be a homotopy of fibered isomorphisms, s.t. $bs\, \phi^t = id|_W$.
Then for every A-directed embedding $f_0 : V \longrightarrow W$, there exists an isotopy $f_t : V \longrightarrow W$, s.t. f_1 is an $\tilde{A}$-directed embedding, where $\tilde{A} := \phi^1_* A$. Moreover, such an isotopy f_t can be chosen arbitrarily C^0-close to the constant isotopy.

<u>proof</u>: We define a sequence of maps $\Phi^{(i)} = \Phi^{t_i}$, $0 = t_0 < t_1 < \_\_\_ < t_N = 1$, s.t. the angle between $\Phi^{(i)}(L)$ and $\Phi^{(i+1)}(L)$ is less than $\frac{\pi}{4}$ $\forall L \in Gr_n(W)$.
Set $A_i := \phi^{(i)}_*(A)$ & consider a tubular nbhd. X of $f_0(V)$, s.t.
(Lemma 1) $R^X_{A_1}$ is open & ample $\rightsquigarrow$ convex integration to formal sol.
$F^{(1)} = \Phi^{(1)}_*(df_0(V))$ $\Longrightarrow$ get embedding f_{t_1} as genuine sol., which is A_1-dir.
$\rightsquigarrow$ repeat inductively for tubular nbhd. X_1 of $f_{t_1}(V)$ ...
$\Longrightarrow$ we get $f_{t_N} = f_1$, which is $\tilde{A}$-directed embedding
(approximation property follows from possibility to approximate at each step) □

Now let's assume the setting of thm. 2 and equip W with a Riemannian metric. We then „cover-up the homotopy G_t by a homotopy of fiber-wise isomorphism", i.e. we just define the fiber-wise isom. ϕ^t by

$$G_t : V \longrightarrow Gr_n(W) \text{ with underlying } g_t : V \longrightarrow W$$

$$\underset{\substack{\| \\ \phi_t(g_t(v))\, G_0(v)}}{G_t(v)} \subseteq T_{g_t(v)} W$$

$\rightsquigarrow$ fiber-wise isom., if we fix $g_t(v) = f_0(v)$

Then we just apply Lemma 2, which yields the claim $\square$

One meaningful application of this is the following:

<u>Corollary (real embeddings):</u> Let (W, J) be an almost complex mfd.

Then the following hold:

a) Every embedding $f_0 : V \longrightarrow (W, J)$, whose tangential lift $G_0 = G(df)$ is homotopic over embeddings to a map
$$G_1 : V \longrightarrow A_{real} \subset Gr_n(W), \quad (\text{resp. } G_1 : V \longrightarrow A_{co\text{-}real})$$
can be isotoped to a real (co-real) embedding $f_1 : V \longrightarrow W$.

b) Let $f_t : V \rightarrow (W, J)$ be an isotopy which connects two real (resp. co-real) embeddings f_0 & f_1. Suppose there exists a family of real (resp. co-real) homomorphisms $F_t : TV \longrightarrow TW$ which covers the isotopy f_t s.t. $df_t \sim F_t$ are homotopic via families of monomorphisms fixed at $t = 0, 1$. Then there exists an isotopy of real (resp. co-real) embeddings
$$\tilde{f}_t : V \longrightarrow W,$$
which connects $\tilde{f}_0 = f_0$ with $\tilde{f}_0 = f_0$, C^0-close to f_t and
$$d\tilde{f}_t \sim F_t \text{ homotopic via families of monomorphisms at } t = 0, 1.$$