Seminar announcement

General information

Instructor: Prof. Chris Wendl
HU Institute for Mathematics (Rudower Chaussee 25), Room 1.301
`wendl@math.hu-berlin.de`
Office hour: Thursdays 14:30–15:30

Seminar webpage: `http://www.mathematik.hu-berlin.de/~wendl/Sommer2025/hPrinzip/`

Moodle: `https://moodle.hu-berlin.de/course/view.php?id=133149`
enrollment key: `flexible`

Time and place: Fridays 11:15–12:45 in 1.114 (Rudower Chaussee 25)

Language: The seminar will be run in English by default, but talks can also occasionally be given in German, depending on the preferences of the participants.

Prerequisites: Contents of the HU's courses *Differentialgeometrie I* and *Topologie I*; some additional knowledge of algebraic topology (as in *Topologie II*) may be helpful for context but is not essential.
All participants will be assumed to be comfortable with smooth n-dimensional manifolds, tensors, differential forms, Riemannian metrics and vector bundles, as well as basic notions from topology such as homotopy equivalence and the fundamental group.

Requirements: All (students and otherwise) are welcome to attend the seminar and may volunteer to give talks, though priority will be given to Bachelor- and Master-students who need credit for the seminar. For students to receive credit, the requirements are the following:

1. Give at least one of the talks (normally this will mean one 90-minute session, but these sessions could be split up if there are too many volunteers), with careful attention to fitting all necessary material within the given time constraints;
2. Submit clearly readable notes for your talk (you are welcome to TeX them, but handwritten notes are also fine), so that they can be scanned and uploaded to the website by the following Monday;
3. Attend the seminar regularly (at most three absences in the semester, barring exceptional circumstances).

Overview

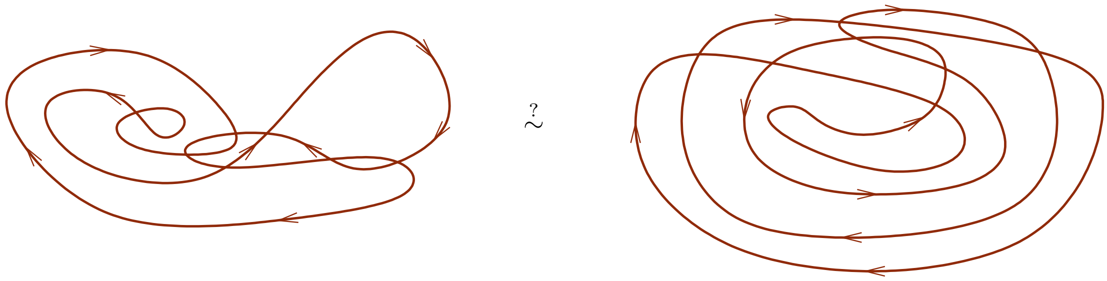
Many natural problems in differential geometry and topology—for example the existence of immersions, symplectic forms, or isometric maps—can be formulated in terms of partial differential relations (PDRs), i.e. equations or inequalities that constrain the partial derivatives of a map. In solving such problems, there is typically a much easier problem that must be solved first: the classification of *formal solutions*, maps which satisfy a corresponding algebraic relation without any constraints on derivatives. To illustrate the idea, consider the following simple problem:

Question: *What conditions are necessary and sufficient for two smooth immersions* $\gamma : S^1 \looparrowright \mathbb{R}^2$ *to be "regularly" homotopic, i.e. related to each other via a smooth* 1*-parameter family of immersions?*

The PDR in this example is the condition that all loops $\gamma : S^1 \to \mathbb{R}^2$ under consideration should be immersions, meaning $\dot{\gamma}(t) \neq 0$ for all $t \in S^1$. If two such loops $\gamma_1, \gamma_2 : S^1 \looparrowright \mathbb{R}^2$ are regularly homotopic, then the corresponding loops of tangent vectors $\dot{\gamma}_1, \dot{\gamma}_2 : S^1 \to \mathbb{R}^2 \setminus \{0\}$ must obviously be homotopic in the punctured plane, which (using the fact that $\pi_1(\mathbb{R}^2 \setminus \{0\}) = \pi_1(S^1) = \mathbb{Z}$) is true if and only if they have the same winding number,

$$\text{wind}(\dot{\gamma}_1) = \text{wind}(\dot{\gamma}_2) \in \mathbb{Z}.$$

Of course, most loops in an arbitrary homotopy from $\dot{\gamma}_1$ to $\dot{\gamma}_2$ will not be actual loops of tangent vectors to immersions $S^1 \looparrowright \mathbb{R}^2$, thus a formal solution seems at first quite far from being an answer to the question that was actually asked. But it is a remarkable fact, known as the *Whitney-Graustein theorem*, that formal solutions to this problem imply genuine solutions: two immersed loops in $\mathbb{R}^2$ are regularly homotopic if and only if the winding numbers of their first derivatives are the same. The condition on winding numbers is generally quite easy to check, even though for a given pair of immersed loops, it may seem impossible in practice to find an explicit regular homotopy between them!



In general, we say that a differential-geometric problem based on a PDR **satisfies the h-principle** if it can be reduced to the problem of finding formal solutions. The latter is typically a matter of standard algebraic topology, with no need for deeper analytical or geometric techniques—as a topological problem it may be easy or hard, but on the surface it is at least much simpler than solving a PDE or PDR. The fact that some geometric problems are "flexible" in this sense was first recognized by Gromov around 1970, though various seemingly unrelated examples of it had been known since the 1950's or earlier. The h-principle comes in a variety of flavors and has applications in a wide range of subjects. Famous examples include the following:

- The Smale-Hirsch immersion theory in differential topology: this is a generalization of the Whitney-Graustein theorem and also includes the notoriously counterintuitive fact that the 2-sphere can be "turned inside out" by a smooth family of immersions $S^2 \looparrowright \mathbb{R}^3$! This is known as the *Smale eversion.* (For a very entertaining and informative video about it, see `http://www.youtube.com/watch?v=wO61D9x6lNY`.)
- The Nash-Kuiper C^1-isometric embedding theorem: among other things, this implies that any Riemannian n-manifold admits C^1-smooth isometric embeddings into arbitrarily small balls in $\mathbb{R}^{n+1}$. This cannot be done with C^2-smooth embeddings; curvature prevents it.
- Existence and uniqueness (up to homotopy) of symplectic and contact structures on open manifolds.

Some more advanced applications include results in foliation theory and the existence of metrics with negative curvature, as well as several groundbreaking advances in symplectic and contact topology involving flexibility of certain classes of contact structures, Legendrian submanifolds and Stein manifolds. A common feature of many h-principles is that they are very hard to visualize, and for this reason it tends to seem surprising whenever an h-principle holds. On the other hand, when the h-principle fails, it often indicates the existence of interesting geometric invariants that contain more than purely topological information.

The goal of this seminar will be to learn the basics of the subject, including at least the three classic applications listed above, and two powerful methods for proving fairly general h-principles: holonomic approximation and convex integration theory. There will be a special emphasis on topics in symplectic and contact geometry, though not exclusively.

Literature

The main portion of the seminar will follow:

- Kai Cieliebak, Yakov Eliashberg and Nikolay Mishachev, *Introduction to the h-principle*, second edition, AMS 2024
  This is a very good book for learning the essentials of the subject, with particularly good coverage of the applications to symplectic geometry. In addition to a detailed discussion of convex integration, it introduces the method of holonomic approximation, which simplifies the proofs of results that previously required the covering homotopy method.

Here are some additional sources that may be worth looking at:

- Hansjörg Geiges, *h-principles and flexibility in geometry*, Memoirs of the AMS, 2003
  A very short book that provides an introduction to the basic ideas, focusing on the covering homotopy method and its applications, and a sketch of convex integration.
- Mikhael Gromov, *Partial differential relations*, Springer 1986
  This is the bible of the subject and includes a large amount of interesting material, but is a difficult read for beginners.
  (online access available via the HU library)
- Vincent Borelli, lecture notes on convex integration (from a 2012 workshop in Les Diablerets), available at `http://math.univ-lyon1.fr/~borrelli/Diablerets/`

Tentative plan of topics

The schedule of talks and assignment of topics will be decided at the end of the introductory meeting on April 25; the following plan is tentative and subject to change. We will work through portions of the book by Cieliebak-Eliashberg-Mishachev.

1. General introduction
2. Jet bundles and PDRs, general formulations of the h-principle (portions of chapters 1, 6 and 7)
3. Holonomic approximation theorem, part 1 (chapter 3)
4. Holonomic approximation theorem, part 2 (chapter 3)
5. The h-principle for open Diff-invariant relations (chapter 8)
6. The Smale-Hirsch immersion theorem and other applications to closed manifolds (chapter 9)
7. Basics on symplectic and contact geometry (chapter 17)
8. Symplectic and contact structures on open manifolds (chapter 18)
9. Microflexibility and isotropic immersions (portions of chapters 21 and 22)

10. Lagrangian and Legendrian immersions (portions chapters 23 and 24)
11. One-dimensional convex integration (chapter 25)
12. The h-principle for ample differential relations (chapter 26)
13. Directed immersions and embeddings (chapter 27)
14. The Nash-Kuiper theorem on isometric immersions (chapter 29)