Symplectic field theory
Problem set 5

Gerard Bargalló i Gómez and Chris Wendl

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This sheet means to achieve two things: studying spaces of almost complex structures and proving some results that we will need soon; and getting some familiarity with holomorphic curves in symplectizations (of contact manifolds) and symplectic cobordisms.

Problem 1. Almost complex structures. Our goal is to study the space of almost complex structures on a given manifold.[1] We start with the linear model: the space of complex structures on $\mathbb{R}^{2n}$, denoted $\mathcal{J}(\mathbb{R}^{2n})$.

a. Show that $\mathcal{J}(\mathbb{R}^{2n})$ is a $2n^2$-dimensional submanifold of $\operatorname{End}_{\mathbb{R}}(\mathbb{R}^{2n})$ that can be presented as a homogeneous space $\mathcal{J}(\mathbb{R}^{2n}) = \mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$ and that its tangent space can be given by $T_J\mathcal{J}(\mathbb{R}^{2n}) = \overline{\operatorname{End}}_{\mathbb{C}}(\mathbb{R}^{2n}, J)$.[2]

   Hint: Consider the action of $\mathrm{GL}(2n,\mathbb{R})$ on $\mathcal{J}(\mathbb{R}^{2n})$ given by $A \cdot J = AJA^{-1}$.

b. Deduce that it has two connected components and that the one containing J_0, denoted $\mathcal{J}^+(\mathbb{R}^{2n})$, is the space of complex structures compatible with the standard orientation of $\mathbb{R}^{2n}$. Convince yourself that $\mathcal{J}^+(\mathbb{R}^{2n}) = \mathrm{GL}^+(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$ is homotopy equivalent to $\mathrm{SO}(2n)/\mathrm{U}(n) = \mathcal{J}^+(\mathbb{R}^{2n}) \cap \mathrm{SO}(2n)$.

c. * To exemplify the subtlety in the topology of this space compute homotopy type of $\mathcal{J}^+(\mathbb{R}^4)$.

   Hint: Show that $\mathcal{J}^+(\mathbb{R}^4) \cap \mathrm{SO}(4)$ is homotopy equivalent to $\mathbb{S}^2$.

d. * Show that the map

$$Y \in \overline{\operatorname{End}}_{\mathbb{C}}(\mathbb{R}^{2n}, J) \mapsto (\mathbb{1} + \frac{1}{2}JY)^{-1}J(\mathbb{1} + \frac{1}{2}JY) \in \mathcal{J}(\mathbb{R}^{2n}) \tag{1}$$

   is well defined near $0 \in T_J\mathcal{J}(\mathbb{R}^{2n}) = \overline{\operatorname{End}}_{\mathbb{C}}(\mathbb{R}^{2n}, J)$, where its derivative is the identity, and it diffeomorphically identifies a neighbourhood of $0 \in T_J\mathcal{J}(\mathbb{R}^{2n})$ with a neighbouthood of $J \in \mathcal{J}(\mathbb{R}^{2n})$.

   Hint: Consider a first order approximation of e^{tY} for $Y \in \overline{\operatorname{End}}_{\mathbb{C}}(\mathbb{R}^{2n}, J)$.

Using the first item we can identify $\mathcal{J}(E)$, the space of complex structures on a vector bundle $E \to M$ and with a given complex structure J, with the space of sections of the fiber bundle $\operatorname{End}_{\mathbb{R}}(E)/\operatorname{End}_{\mathbb{C}}(E, J)$. Recall that if $E = TM$ then $\mathcal{J}(M) := \mathcal{J}(TM)$ is called the space of almost-complex structures on M.

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[1] Recall that a complex structure on a vector space is an endomorphism that squares to $-\mathbb{1}$. Similarly, a complex structure on a vector bundle $E \to M$ is a bundle endomorphism that squares to $-\mathbb{1}$ (i.e. a fiber-wise complex structure). When $E = TM$ complex structures are called almost-complex structures to differentiate them from complex structures, i.e. the *integrable* almost-complex structures (for which M would be a complex manifold).

[2] Note that the same reasoning exhibits the space of (linear) symplectic structures on $\mathbb{R}^{2n}$ as the homogeneous space $\mathrm{GL}(2n,\mathbb{R})/\mathrm{Sp}(2n)$. A consequence of this and that (as we mostly saw in sheet 3) the symplectic and complex general linear groups deformation retract into the unitary group is that the space of complex structures and symplectic structures on a vector space are homotopy equivalent. This means that the space of complex structures on a vector bundle E and the space of symplectic structures on E (bilinear, skew-symmetric, non-degenerate bundle pairings) are homotopy equivalent. Note that when $E = TM$ these are "almost-symplectic structures" as they need not be closed.

e. Explore the topology of $\mathcal{J}(\mathbb{T}^4)$, the space almost complex structures on the 4-torus, which intricate (find a good heuristic to see it has many connected components).

   Hint: Study $\mathcal{J}^+(\mathbb{T}^4)$ using the computation of $\mathcal{J}^+(\mathbb{R}^4)$ done before.

f. Show that $\mathbb{S}^4$ does not admit any almost-complex structure, $\mathcal{J}(\mathbb{S}^4) = \varnothing$.

   Hint: Throwback to the first problem of the first sheet.

Remark. The existence of almost complex structures is moderately subtle as far as I know. A necessary condition is that the second Stiefel-Whitney class admits an integral lift (the putative c_1). In dimension 4 we also know that the Hirzebruch signature formula must be satisfied. A celebrated theorem of Wu states that these two necessary conditions are sufficient in dimension 4. No such strong result is known in general. Note that both $\mathbb{S}^2$ and $\mathbb{S}^6$ admit almost complex structures (essentially by seing them as the units spheres in the imaginary part of the quaternions and octonions respectively and using their algebra structure.) These (and $\mathbb{S}^0$ trivially) are the only spheres that admit an almost complex structure. Calabi actually showed that none of the almost-complex structures on $\mathbb{S}^6$ are integrable: if $\mathbb{S}^6$ has an *integrable* complex structure is completely open. Other than this I would say that the most general known result is the characterization of the existence of *stable* almost complex structures. A manifold M is said to admit a stable almost complex structure if there is a complex structure on the bundle $TM \oplus \mathbb{R}^k$ for some $k \geq 0$. A necessary condition, once more, is that the second Stiefel-Whitney class admits an integral lift: in this case this is also sufficient.[3] For example, $\mathbb{S}^4$ is stably complex.

Problem 2. Tame and compatible almost-complex structures. We now turn to the space of almost complex structures that play nice with the symplectic form. Let (M, ω) be a symplectic manifold and J an almost complex structure on it. We say it is *tame* (or tamed by ω) if ω is positive on complex lines (i.e. $\omega(X, JX) > 0$ for all $p \in M$, $X \in T_pM \setminus 0$) and *compatible* with ω if moreover we also have that $g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ defines a Riemannian metric. The spaces of such almost complex structures are denotes $\mathcal{J}_\tau(M, \omega)$ and $\mathcal{J}(M, \omega)$ respectively.

a. Check that $J \in \mathcal{J}(M)$ is compatible with ω if and only if its ω-tame and ω is J-invariant. Check as well that the $\mathbb{C}$-valued pairing $h = g_J + i\omega$ is a Hermitian bundle metric.

As before, let us first study the linear case and show that there is a unique compatible almost complex structure up to homotopy, which we see in three different ways (one of these should be asterisc, choose your fighter). The tame case we adjourn to the end of the exercise.

b. Show that $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ is a $n(n+1)$-dimensional submanifold of $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2n})$ that can be presented as a homogeneous space $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathrm{std}}) = \mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$ and that its tangent space can be given by $T_J\mathcal{J}(\mathbb{R}^{2n}) = \{Y \in \overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{R}^{2n}, J) : \omega(Y\cdot,\cdot) + \omega(\cdot, Y\cdot) = 0\}$. Deduce that $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ is contractible.

c. Identify $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ with the space of positive-definite symmetric symplectic $2n \times 2n$ matrices. Deduce that $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ is contractible.

d. Show that the map $J \mapsto g_J$ defines a homotopy equivalence between $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ and the space of bilinear symmetric and nondegenerate pairings on $\mathbb{R}^{2n}$ (i.e. inner products). Note that this space is a convex open subset of a vector space, deduce that $\mathcal{J}(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ is contractible.

   Hint: Construct a continuous left-inverse to this map via polar decomposition.

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[3] This has to do with the fact that the obstructions of the non-stable problem are controlled by the homotopy type of $\mathrm{SO}(2n)/\mathrm{U}(n)$, which are at least as subtle as those of $\mathbb{S}^n$. The stable problem, however, is controlled by the limit $\lim_{\to n} \mathrm{SO}(2n)/\mathrm{U}(n)$, whose homotopy groups are much better behaved due to Bott periodicity.

Now we globalize these results.[4] Here are two results that will come in handy and you can use as a black-box: a Serre fibration with path-connected base and weakly contractible fibers is a weak homotopy equivalence; the space of sections of a smooth (locally trivial) fiber bundle *with contractible fibers* on a smooth manifold is non-empty and contractible.

e. Express $\mathcal{J}(M, \omega)$ as the space of sections of a fiber bundle (see Problem 1) and conclude (from part *b.* or *c.*) that it is non-empty and contractible.

f. Show that $\mathcal{J}(M, \omega)$ is homotopy equivalent to the space of Riemannian metrics via the map $J \mapsto g_J$. Conclude that $\mathcal{J}(M, \omega)$ is non-empty and contractible.

g. * Fix $J_0 \in \mathcal{J}(M, \omega)$, it will be fundamental for us to understand the local structure of this space near J_0. Convince yourself that

$$T_{J_0}\mathcal{J}(M, \omega) = \{Y \in \Gamma(M, \mathrm{End}(TM)) | \, YJ_0 + J_0Y = 0, \;\; \omega(Y\cdot, \cdot) + \omega(\cdot, Y\cdot) = 0\}.$$

For $Y \in T_{J_0}\mathcal{J}(M, \omega)$ close enough to 0 define the **exponential map** as follows

$$\exp_{J_0} Y := (\mathbb{1} + \frac{1}{2}J_0Y)^{-1}J_0(\mathbb{1} + \frac{1}{2}J_0Y).$$

Its derivative at 0 is the identity and it diffeomorphically parametrizes a neighbourhood of $J_0 \in \mathcal{J}(E, \omega)$ by a neighbourhood of $0 \in T_{J_0}\mathcal{J}(E, \omega)$.

h. Assume that now (M, α) is a strict contact manifold and consider $\mathcal{J}^{\mathrm{SFT}}(\alpha)$ the SFT-adapted almost-complex structures (translation invariant, $J\partial_r = R_\alpha$ and preserves ξ where it is compatible with $d\alpha$). Similarly compute $T_{J_0}\mathcal{J}^{\mathrm{SFT}}(\alpha)$.[5]

So far we have shown that the space of tame almost complex structures is non-empty. It is also contractible:

i. Bonus: Consider $X(\mathbb{R}^{2n})$ to be the space of pairs (ω, J), where ω is a non-degenerate skew-symmetric bilinear form on $\mathbb{R}^{2n}$ and $J \in \mathcal{J}(\mathbb{R}^{2n}, \omega)$. Similarly define $X_\tau(\mathbb{R}^{2n})$ as the space of pairs (ω, J) where ω is as before and $J \in \mathcal{J}_\tau(\mathbb{R}^{2n}, \omega)$. Consider the diagram

$$\begin{tikzcd}
X(\mathbb{R}^{2n}) \arrow[rr, hook] \arrow[dr] & & X_\tau(\mathbb{R}^{2n}) \arrow[dl] \\
& \mathcal{J}(\mathbb{R}^n) &
\end{tikzcd}$$

given by the obvious projections and inclusion. Show that the vertical arrows are weak homotopy equivalences. Deduce that $\mathcal{J}_\tau(\mathbb{R}^{2n}, \omega)$ must be contractible.

j. Bonus: Deduce that $\mathcal{J}_\tau(M, \omega)$ is non-empty and contractible.

Remark. That $\mathcal{J}(M, \omega)$ is non-empty and contractible says that symplectic manifolds have a unique compatible (or tame) almost-complex structure, up to homotopy. Then, we can study symplectic geometry via complex geometry up to homotopy. Holomorphic curves behave (somewhat) well under (generic) homotopies of almost complex structures and so they give useful invariants of symplectic manifolds. The description given here of $T_J\mathcal{J}^{SFT}$ will be important when we study this characterization. Finally, note the phylosophical role of the 2-out-of-3 property ($O \cap S = S \cap G_{\mathbb{C}} = O \cap G_{\mathbb{C}} = U$) in these problems.

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[4] We globalize them to the tangent bundle TM and the symplectic form ω. A moment's thought shows that this works for a vector bundle with a symplectic form (in the general sense, it needs not be closed, which does not even make sense).

[5] You can also do it for stable Hamiltonian structures, nothing changes for this computation.

Problem 3. Holomorphic curves in symplectizations of contact manifolds. Consider a (strict) contact manifold (M, α) and its symplectization $(\widehat{W} = \mathbb{R} \times M, \omega = d(e^r\alpha))$. All almost complex structures are understood to be SFT adapted almost-complex structures $\mathcal{J}^{\mathrm{SFT}}$. We write u as short-hand for $u : (\Sigma \setminus \Gamma, j) \to (\widehat{W}, J)$ be a pseudo-holomorphic curve.

a. Given a closed Reeb orbit γ of period T, we define the trivial cylinder $u_\gamma : \mathbb{R} \times \mathbb{S}^1 \to \widehat{W}$ by $u_\gamma(s, t) = (Ts, \gamma(Ts))$. Verify that this is a J-holomorphic curve (for all J SFT-adapted almost-complex structure). Assuming that γ is non-degenerate, compute the index[6] of u_γ.

b. Let u be a J-holomorphic curve as above and consider the integral $\int_{\Sigma\setminus\Gamma} u^*\omega$, the "naive" notion of energy. Show that the integral is infinite for the trivial cylinder $u = u_\gamma$. Consider now $E(u)$ the Hofer energy (as in the lecutres). Show that it is finite for u_γ, non-negative in general and zero if and only if u is constant.

c. * The contact energy of a pseudo-holomorphic curve u is defined as

$$E_\alpha(u) = \int_{\Sigma\setminus\Gamma} u^* d\alpha.$$

Show that E_α is always non-negative and compute $E_\alpha(u)$ in terms of the periods of the asymptotic Reeb orbits. Show that if $E_\alpha(u) = 0$, the curve u must be everywhere tangent to the distribution spanned by ∂_r and the Reeb vector field R_α.

d. * Conclude that there are no closed holomorphic curves in symplectizations and that any pseudo-holomorphic curve in a symplectization must have at least one positive puncture.

e. Consider a pseudo-holomorphic curve $u : (\Sigma \setminus \Gamma, j) \to (\widehat{W}, J)$ and a closed Reeb orbit γ such that: u has a m positive punctures, where it is asymptotic to the closed Reeb orbits $\gamma^{j_1}, \dots, \gamma^{j_m}$ (all different iterates of the orbit γ); and it has r negative punctures where it asymptotes orbits $\gamma^{k_1}, \dots, \gamma^{k_r}$ (all different iterates of γ). If $\sum_{i=1}^m j_i = \sum_{i=1}^r k_i$, show that any such holomorphic curve u is a multiple cover of the trivial cylinder u_γ.

f. * Show that the real part of a holomorphic curve satisfies the maximum principle. More precisely, if $u : (\Sigma \setminus \Gamma, j) \to (\widehat{W}, J)$ is a pseudo-holomorphic curve and $u_\mathbb{R} = pr_\mathbb{R} \circ u$ satisfies the maximum principle. Deduce, once more, that there must always be a positive puncture.

g. If all positive or all negative asymptotic orbits of a pseudo-holomorphic curve are simple, then so is the curve.

h. Bonus: Consider the splitting $T\widehat{W} = \varepsilon \oplus \xi$ where ε is the distribution spanned by ∂_r and R_α. Prove the points of tangency of ε and a pseudo-holomorphic curve u are either all of u or only finitely many.

   Hint: Observe that if u is a J-holomorphic curve, so is $u + a$ for all $a \in \mathbb{R}$. Linearize this and use the similarity principle.

Remark. This problem should start to justify why we use the notions of energy that we do and why we have defined SFT-almost-complex structures in this way. Ultimately, we will see that finite energy implies being asymptotic to Reeb orbits (holomorphic curves detect Reeb orbits), which *c.* is a key point for. Hofer was the first to recognize the importance of holomorphic curves with punctures when he used them to prove the Weinstein conjecture in some cases: he leveraged the

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[6] The index of a curve u will be the dimension of the moduli space in which u lives, we define it as follows: if $u : (\dot{\Sigma} = \Sigma \setminus (\Gamma_+ \cup \Gamma_-), j) \longrightarrow (\widehat{W}, J)$ positively/negatively asymptotic to the non-degenerate orbits $\gamma_1^\pm, \dots, \gamma_{\ell_\pm}^\pm$, then

$$\operatorname{ind} u := (n-3)\chi(\dot{\Sigma}) + 2c_1^\tau(u^*T\widehat{W}) + \sum_{i=1}^{\ell_+} \mu_{\mathrm{CZ}}^\tau(\gamma_i^+) - \sum_{i=1}^{\ell_-} \mu_{\mathrm{CZ}}^\tau(\gamma_i^-).$$

fact that overtwisted discs and spheres (with elliptic points) give rise to holomorphic discs that can then be continued until they must break a finite-energy holomorphic plane, therefore detecting a contractible Reeb orbit.

Finally, we note that the contact energy proof that holomorphic curves must have at least one positive end generalizes to completed symplectic cobordisms and that the maximum principle one generalizes to Stein cobordisms (but *not* exact ones).

Problem 4. Some holomorphic curves in irrational ellipsoid cobordism. Consider the trivial ellipsoid fillings

$$E(a,b) = \{(z_1,z_2) \in \mathbb{C}^2 \mid \frac{\pi}{a}|z_1|^2 + \frac{\pi}{b}|z_2|^2 \leq 1\}.$$

We have previously studied the contact manifolds $\partial E(a,b)$ and seen that, when a/b is irrational, the natural contact form is non-degenerate and there are only two (embedded) orbits γ_1 and γ_2 (whose image coincides with $\gamma_j = \partial E(a,b) \cap \{z_{j+1} = 0\}$, where $2+1=1$).

a. Show that the completion $\widehat{W}$ can be identified with $\mathbb{C}^2$ and that the standard contact structure i on $\mathbb{C}^2$ is an SFT-admissible almost complex structure for $\widehat{W}$.

b. * Show that the moduli spaces of i-holomorphic planes (once punctured spheres) asymptotic to γ_1 (resp. γ_2) are non-empty. Compute the index (defined in part *a.* of the previous exercise) of the holomorphic curves you have found.

Take $c \leq a$ and $d \leq b$, then $E(c,d) \subseteq E(a,b)$ and consider the symplectic cobordism W given by $E(a,b) \setminus E^\circ(c,d)$. We require both a/b and c/d to be irrational and denote the two Reeb orbits of $\partial E(a,b)$ by γ_j^+ and the two of $\partial E(c,d)$ by γ_j^-.

c. * Similarly show that the moduli space of i-holomorphic cylinders positively asymptotic to γ_1^+ (resp. γ_2^+) and negatively asymptotic to γ_1^- (resp. γ_2^-) is non-empty. Compute the index of these curves you have found.

d. Let k be a positive integer, find i-holomorphic curves positively asymptotic to $(\gamma_1^+)^k$ (resp. $(\gamma_2^+)^k$) and negatively asymptotic to $(\gamma_1^-)^k$ (resp. $(\gamma_2^-)^k$) is non-empty. Compute the index of these curves you have found.

Remark. This problem explores holomorphic curves in the symplectization of the sphere and in trivial sphere cobordisms for many non-degenerate contact forms (that define the same contact structure). You should play with the parameters a,b,c,d to see how the behavior of the holomorphic curves change. Explore the different values of the index you can get and therefore the different expected dimensions (or lack there of) the moduli space of holomorphic curves. Here are two things to watch out for: when iterating orbits curves are can be expected to disappear; that varying a,b or c,d slightly within irrational ellipsoids does not really change anything while big changes can have drastic effects.

Bonus problem. Holomorphic curves in the symplectization of the sphere. The goal of this exercise is to study holomorphic curves in the symplectization of the sphere with respect to the standard contact form (hence, not a non-degenerate form) and relate them to the theory of holomorphic line bundles over $\mathbb{CP}^1$.

a. Let L be the tautological line bundle over $\mathbb{CP}^1$ and $Z \subseteq L$ the zero section. Identify biholomorphically the space $L \setminus Z$ and the symplectization $\mathbb{R} \times \mathbb{S}^3$ with the complex structure induced from $\mathbb{C}^2 \setminus 0$.

b. A meromorphic section s of L determines a holomorphic curve in $\mathbb{R} \times \mathbb{S}^3$ with positive/negative ends corresponding to the poles/zeroes of s. Conversely, a genus 0 holomorphic curve into $\mathbb{R} \times \mathbb{S}^3$ which intersects each fiber of $L \setminus Z \to \mathbb{CP}^1$ transversely exactly once at a single point, except for the fibers over the Reeb orbits at the positive and negative ends, comes from a meromorphic section of L as described.

   Bonus: what is the analogue for higher genus?

We now want to describe the space of holomorphic spheres in $\mathbb{R} \times \mathbb{S}^3$ with one positive and m negative punctures, which naturally appear in contact homology.

c. Consider the following divisor $D = -kp + \sum_{i=1}^{m} k_i q_i$ on $\mathbb{CP}^1$. Identify the space of meromorphic sections s of L with $[s] = D$ with $\mathbb{C}^*$ when $|D| = -1$ and with $\varnothing$ when $|D| \neq -1$.

   In plain language: show that the meromorphic sections of L with a pole at p of order k and zeroes at $q_1, \dots, q_m$ of order $k_1, \dots, k_m$ (all points are distinct) come in $\mathbb{C}^*$-families when $\sum_{i=1}^{m} k_i = k-1$ and don't exist otherwise.

d. Let γ and $\gamma_1, \dots, \gamma_m$ be pair-wise distinct simple Reeb orbits in $\mathbb{S}^3$ (i.e. Hopf fibers). Describe the space $\mathcal{M}_0(\gamma^k; \gamma_1^{k_1}, \dots, \gamma_m^{k_m})$ of holomorphic spheres in $\mathbb{R} \times \mathbb{S}^3$ with one positive asymptotic to γ^k and m negative punctures asymptotic to $\gamma_1^{k_1}, \dots, \gamma_m^{k_m}$.

e. Let $U, U_1, \dots, U_m$ be small Reeb invariant open subspaces of $\gamma, \gamma_1, \dots, \gamma_m$ (meaning open sets of nearby Reeb orbits). Describe the space of holomorphic spheres in $\mathbb{R} \times \mathbb{S}^3$ with one positive asymptotic to γ^k and nearby orbits in U and m negative punctures asymptotic to $\gamma_1^{k_1}, \dots, \gamma_m^{k_m}$ and nearby orbits in $U_1, \dots, U_m$. If we let the open sets be large, what can you say of the structure of the moduli space?

Remark. These last two problems revolve around holomorphic curves in symplectizations/cobordisms for *different* contact forms that define the same contact structure. The first explores the non-degenerate case, while the second is a first contact with the Morse-Bott case. One could ask what happens when we take the Morse-Bott form on $\partial E(1,1)$ explored in the last problem and we modify it every so slightly to an irrational ellipsoid $\partial E(1+\varepsilon, 1-\varepsilon)$, explored in the second to last problem. This new ellipsoid will have two orbits, the holomorphic curves between these will then be given by the gradient flow lines of a Morse function on $\mathbb{P}^1$ with two critical points (one on each point representing an orbit), which come in one-parameter families.