Symplectic field theory
Problem set 8

Gerard Bargalló i Gómez and Chris Wendl

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This problem set (as the upcoming ones will be) is centered around specific theorems that we find beautiful; it is finally time to enjoy the fruits of all of the theory building. In fact, these sheets should be understood as gentle guided proofs of theorems. Consequently, there is naturally a lot of text to establish certain ideas, tools or side-theorems. There will be only a small amount of gray-boxing, as some tools will be stated in great generality and proved in concrete cases, but these exercises should constitute decently self-contained theorem proofs.

The first part starts with some examples to familiarize ourselves with *closed* holomorphic curves in 4-manifolds. On the second part we state the theorems to be proven and from then on the other parts assemble all the machinery needed to prove them (positivity of intersections and adjunction, automatic transversality and Gromov compactness).

PART I: Examples Let us set convention and terminology about a few things. All manifolds here will be orientable. Closed (connected) 4-manifolds will usually by denoted by M. When we talk about the index $\operatorname{ind} u$ of a genus g J-holomorphic curve u representing $A = [u]$, we mean the virtual dimension of the moduli space $\mathcal{M}_g(A, J)$. When we talk about the self-intersection of a curve in the class $A \in H_2(M; \mathbb{Z})$, or the intersection of two curves in the classes A and B, we speak of $A \cdot A$ and $A \cdot B$, where $- \cdot -$ represents the $\mathbb{Z}$-valued intersection pairing in (the torsion-free part of) $H_2(M; \mathbb{Z})$. If two classes A and B are represented by 2-dimendional closed submanifolds C_A and C_B, then $A \cdot B = \# C_A \cap C_B$, possibly after a small perturbation of one of them to achieve transversality (so there are finitely many intersections). In particular, the self-intersection of an embedded curve is the count of intersection points of the curve with a transverse push-off of it.

The fiber classes of ruled surfaces. Here is the simplest relevant example of a symplectic fibration, which will be one of the protagonists in this sheet.

1. Consider the symplectic manifold $(\mathbb{S}^2 \times \mathbb{S}^2, \sigma \oplus \sigma)$, where σ is the standard area form on $\mathbb{S}^2$. Regard it as a trivial fibration by spheres $\pi : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{S}^2$ by projecting to the first component. Show that any fiber F_1 is a symplectic submanifold, of self-intersection 0, and a holomorphic curve for the complex structure $i \oplus i$ (here i is the standard complex structure on $\mathbb{S}^2$) and its index is 2. Notice that the situation is symmetric for F_2, the fiber with respect to the other coordinate projection $\mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{S}^2$, and so there are two symplectic spheres F_1 and F_2 of self-intersection 0 intersecting transversely and positively exactly once. See the figure left.

In general, a *symplectic fibration* on a closed symplectic manifold (M, ω) is a fiber bundle $\pi : M \to B$ (to a closed base) such that ω restricts as a non-degenerate 2-form on the fibers (so fibers are symplectic submanifolds).

2. Let (M, ω) be a symplectic 4-manifold with a symplectic fibration $\pi : M \to \Sigma_b$ with fibers diffeomorphic to Σ_f (here f and b are the genus of the fiber/base surfaces). Show that the self-intersection of the fibers is 0 and that, with respect to some compatible almost complex structure making the fibers holomorphic, their index is $2 - 2f$.

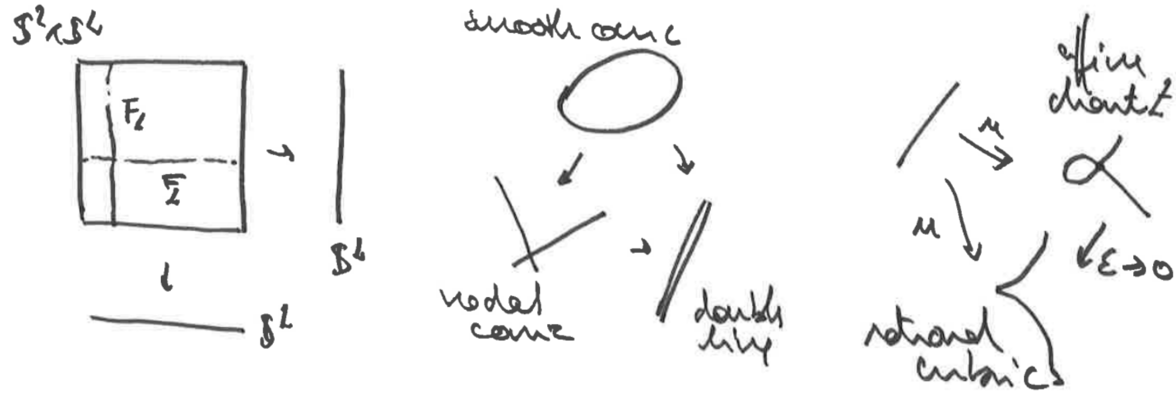

Figure 1: Left: 1. Center 4. Right: Bonus 1.

Note that a symplectic fibration can be topologically trivial but not symplectically: a symplectic form on a product may not be a product symplectic form.[1] We want to study **symplectically ruled 4-manifolds**, which are symplectic 4-manifolds with a symplectic fibration with genus 0 fibers. Classifying surfaces in 3-space that are sweapt by straight lines goes back two centuries, and the equivalent in projective geometry about a century (I think).

Notice that since all surfaces are symplectic, their product is also symplectic with the split symplectic form. One can still make sense of a twisted version of this statement which we mention now even if it will not be needed in this sheet. (A consequence of) Thurston's trick states that *every smooth oriented $\mathbb{S}^2$-bundle over a closed surface admits admits a unique deformation class of symplectic structures for which it becomes a symplectic ruled surface.*[2]

Curves in $\mathbb{CP}^2$. By a degree d projective curve in $\mathbb{CP}^2$ we mean the zero set

$$\{[X,Y,Z] \in \mathbb{CP}^2 |\ P(X,Y,Z) = 0\}$$

of a degree d *homogeneous* polynomial P on $\mathbb{C}^3$. The degree 1, 2 and 3 projective curves are called lines, conics and cubics respectively (one can keep going, of course).

3. Show that the space of lines in $\mathbb{CP}^2$ is isomorphic to $\mathbb{CP}^2$ and that through any two (distinct) points in $\mathbb{CP}^2$ there is a unique line passing through them. Describe any line as a degree[3] 1 genus 0 holomorphic curve with (homogenous) coordinates given by linear functions. Conversely, show that any degree 1 genus 0 holomorphic curve in $\mathbb{CP}^2$ admits such a parametrization. Therefore, $\mathcal{M}_0(H,i) = \mathbb{CP}^2$. Deduce from the fact that any two lines intersect once that the virtual dimension of that moduli space is indeed 4.

4. Show that the space of conics in $\mathbb{CP}^2$ is isomorphic to $\mathbb{CP}^5$ and that through any generic 5 points there is a unique conic. Explain why all conics fall into one of the types in the drawing (middle; covertly 4 types). Can you describe them as degree 2 holomorphic curves (possibly nodal or multiply covered)? Which of these are given by genus 0 holomorphic curves with

[1] In fact, note that there are two $\mathbb{S}^2$ bundles over a closed surface up to diffeomorphism.

[2] The key property is that the fibers are oriented and homologically non-torsion in the total space. Then, some differential topology allows us to choose a closed 2-form on the total space such that it restricts to (essentially) the standard area form on the fiber spheres. This is non-degenerate along the fiber (i.e. vertical) directions but not on base (i.e. horizontal) directions. To fix this we pull-back a symplectic form on the base, a horizontal closed 2-form, and add it to the symplectic form on the fiber $\mathbb{S}^2$. It may not be non-degenerate but rescaling the horizontal part by a sufficiently large constant does the trick.

[3] A holomorphic curve represents a cycle $[u] \in H_2(\mathbb{CP}^2)$, which is identified with $\mathbb{Z}$, and so it has a *degree*. Usually, we denote the class of a line by H and notice that we can identify it with $1 \in \mathbb{Z} = H_2(\mathbb{Z})$.

coordinate functions polynoamials of degree 2? Deduce that the compactified space of degree 2 genus 0 holomorphic curves $\overline{\mathcal{M}}_0(2H, i)$ can be identified with $\mathbb{CP}^5$. Verify that the virtual dimension of that moduli space is indeed 10.

Hint: Some of the behaviour in the picture can be seen from the family of conics $aX^2 + b(Y^2 + Z^2) = 0$ for $[a; b] \in \mathbb{CP}^1$; and if $a \neq 0$, and $c := -b/a$, we can parametrize them by $u_{[a,b]}([s; t]) = [c(s^2 + t^2); s^2 - t^2; 2st]$.

A word of caution: these moduli spaces are manifolds but that does not mean the curves are Fredholm regular; the analysis could be predicting a different manifold structure. Worry not, we will see later that these are in fact automatically Fredholm regular.[4]

Bonus 1. Consider the following family of rational cubics $P_\varepsilon(X, Y, Z) := X^3 + \varepsilon Z X^2 - Z Y^2 = 0$ indexed by $\varepsilon \in \mathbb{R}^{\geq 0}$. Express them as degree 3 genus 0 holomorphic curves: immersed with a single transverse double point for $\varepsilon > 0$ and non-immersed with a critical point for $\varepsilon = 0$ (as in the right figure).

Remark. This should help understand the δ in the adjunction formula, it is 1 for all these. In fact, it is a consequence of the adjunction formula that a projective curve of degree d and (geometric) genus g must satisfy the degree-genus formula (plug in $[u] = dH$ and use the computation of $c_1(H)$ from before):

$$g = \frac{(d-1)(d-2)}{2} - \delta.$$

Bonus 2. Explain how to realize all (possibly singular) projective curves of degree d as holomorphic curves $(\Sigma_g, j) \to (\mathbb{CP}^2, i_{\mathbb{CP}^2})$. Conversely, show that all degree d holomorphic curves into $(\mathbb{CP}^2, i_{\mathbb{CP}^2})$ have components given by polynomials.[5]

PART II: The main theorems In the examples we've seen that a fiber in a symplectically ruled 4-manifold is an embedded symplectic sphere with self-intersection 0 (to be known as a 0-sphere). This would seem to be a local statement, but it's not: our first theorem establishes the converse in the minimal[6] case!

Theorem 1 (Gromov). *Let (M, ω) be a closed minimal symplectic 4-manifold and C a symplectically embedded 2-spheres with self-intersection 0. Then, (M, ω) is symplectically ruled by curves in the class $[C] \in H_2(M)$.*

Since an open neighbourhood of C in M is foliated by curves homologous to C, "local" ruledness is not the most surprising, but that this can be done symplectically and then globally is surprising at first. In the case of the trivial sphere bundle over $\mathbb{S}^2$, we had two 0-curves intersecting positively once. Once more, just the existence of this two curves with prescribed intersection pattern determines the symplectic manifold:

Theorem 2 (Gromov). *Let (M, ω) be a closed minimal symplectic 4-manifold, C_1 and C_2 symplectically embedded 2-spheres, each with self-intersection 0 and intersecting each other transversely and positively exactly once. Then, (M, ω) is symplectomorphic to $(\mathbb{S}^2 \times \mathbb{S}^2, a_1\sigma \oplus a_2\sigma)$, where $a_1 = \omega(C_1)$ and $a_2 = \omega(C_2)$ are the symplectic areas of C_1 and C_2.*

In the course of proving these theorems the following problem will also become apparent: we saw that there is a unique line through every point in $\mathbb{CP}^2$, can we expect this "classical projective

[4]In this case this is directly related to the algebraic notion of positivity of sheaves. Automatic transversality in this setting is just the statement that the positive degree twisting sheaves have vanishing first cohomology.

[5]Hint: Use that the meromorphic functions of $\mathbb{CP}^1$ are the rational functions.

[6]We say that a symplectic manifold is minimal if there are no *exceptional spheres*, i.e. embedded symplectic spheres of self-intersection -1.

geometry" fact to hold with respect to any almost complex structure on $\mathbb{CP}^2$?

Theorem 3 (Gromov). *Let J be any ω_{FS}-compatible almost complex structure on $(\mathbb{CP}^2, \omega_{FS})$. There exists an embedded J-holomorphic sphere through any given point in $\mathbb{CP}^2$.*

Let's sketch the proof of the first theorem, which will unfold over the next exercises:

1. Suppose that (M, ω) is a symplectic manifold and C a smooth 2-dimensional submanifold. Sketch a proof of the following fact: *C is a symplectic submanifold if and only if there exists an ω-tame almost complex structure J preserving TC* (i.e. C is a J-almost complex submanifold).

This means that we can regard C as an element of $\mathcal{M}_0([C], J)$. We then use "automatic transversality" to show that this curve is Fredholm regular, so a neighbourhood $\mathcal{U} \subseteq \mathcal{M}_0([C], J)$ of this curve has manifold structure. We then use adjunction and positivity of intersections to show that each curve in U maps to a J-sphere near C and disjoint from it, in fact foliating a neighbourhood of C (local version of the theorem, local ruledness). We then use compactness properties to show that this open neighbourhood is also closed, so it is the entire M; and that the moduli space is itself compact (here minimality is used).[7] This establishes a map $M \to \mathcal{M}_0([C], J) = \Sigma_b$ by sending each point to the unique J-curve in $\mathcal{M}_0([C], J)$ through it, yielding the desired fibration.

PART III: Positivity of intersection and adjunction formula

Positivity of intersection Consider $A, B \in H_2(M)$ and denote their homological intersection $A \cdot B$. If we have J-holomorphic curves u and v representing those classes (for some J), $A = [u]$, $B = [v]$, and $u \pitchfork v$, then $A \cdot B = \#u \cap v$ and every intersection point contributes positively (consequence of similarity). In chapter 2.6 in the lectures we essentially saw the foundations for a generalization of this: we do not need u and v to be in transverse position to count their intersections. First, they either have the same image or have isolated intersections. Secondly, at each intersection there is an intersection index (essentially an order of tangency) that is at least 1 and 1 if and only if the intersection is transverse. The sum of this index over all intersection points is the homological intersection. This shows:

Theorem (gray box). *Let (M, J) be an almost complex 4-manifold and $u : (\Sigma, j) \to (M, J)$ and $v : (\Sigma', j') \to (M, J)$ be two J-holomorphic curves with non-identical images. Then, they have finitely many intersections, the naive count of which is a lower bound for $[u] \cdot [v]$. Moreover, $[u] \cdot [v] = 0$ if and only if they are disjoint and $[u] \cdot [v] = 1$ if and only if they have a unique transverse positive intersection.*

Adjunction formula Similarly to the above, we can consider the way a holomorphic curve intersects itself. We saw in chapter 2.6 that a simple holomorphic curve intersects itself at finitely many points and has finitely many non-immersed points (i.e. critical points); meaning that it has finitely many singularities. There is also a sensible count of these, which we call $\delta(u)$, where singularities contribute positively (by performing a careful perturbation of $u : (\Sigma, j) \to (M, J)$, we can consider it immersed, and then just count self-intersections with multiplicity). Similarly as before, if $\delta(u) = 0$ then u is embedded, as all singularities count positively. Positive transverse self-intersections of u contribute 1 to δ.

Theorem (Adjuntion formula). *Let (M, J) be an almost complex 4-manifold and $u : (\Sigma, j) \to (M, J)$ a simple J-holomorphic curve. There is a count $\delta(u) \in \mathbb{Z}^{\geq 0}$ that is 0 if and only if u is*

[7] There is a subtle point here, compactness will require J to be generic, so the J in question in this sketch needs to be slightly perturbed, which is not a problem by Fredholm regularity.

embedded and that verifies the following equality

$$[u] \cdot [u] = 2\delta(u) + c_1([u]) - \chi(\Sigma).$$

Notice that it follows from the theorem that if all the somewhere injective (= simple) curves in a moduli space $\mathcal{M}_g(A, J)$ have the same δ, and so if one simple curve is embedded, all of them are in that moduli space.

2. Prove the theorem in the case u is immersed with transverse self-intersections.

3. Show that the index of an embedded J-holomorphic sphere u is given by $\operatorname{ind}(u) = 2+2[u]\cdot[u]$. Deduce that the only embedded spheres that exist for generic J are those of self-intersection $[u] \cdot [u] \geq -1$.

The foliation property of 0-curves Consider a holomorphic curve $u : (\Sigma, j) \to (M, J)$ and write $C = u(\Sigma)$ for its image. Let U be an open neighbourhood of C in M, we say that is foliated by C-curves if there is a unique J-curve homologous to C through every point such that through two points the curves through them are either identical or disjoint. This is the "locally ruled" condition and is guaranteed as soon as:

4. * Consider $u : (\mathbb{S}^2, j) \to (M, J)$ an *embedded* J-sphere of $\operatorname{ind}(u) = 2$. If u is Fredholm regular, then a neighbourhood U of $C = u(\mathbb{S}^2)$ is foliated by the C-curves given by an open neighbourhood $\mathcal{U} \subseteq \mathcal{M}_0([C], J)$ of u.

PART IV: Automatic transversality We now see that there is a topological condition that implies Fredholm regularity that certain curves.

Theorem (Automatic transversality). *Let (M, J) be an almost complex 4-manifold and $u : (\Sigma, j) \to (M, J)$ a immersed J-holomorphic curve. If* $\operatorname{ind} u > -\chi(\Sigma)$*, then u is Fredholm regular.*

5. Prove the $\chi(\Sigma) = 2$ case of this theorem.

   Hint: Consider the linear Chauchy-Riemann operators under the tangent-normal splitting and problem 2.b. in set 4.

PART V: Compactness theory of exceptional and fiber curves It's time to practice the SFT compactness theorem in the puncture-less case with closed target, i.e. Gromov compactness. Bubbling and degenerations of the domain are the only things that can go wrong now. Here is a constraint that will be needed in what follows. Let $(u_k : (\Sigma, j_k) \to (M, J))_k$ be a sequence of J-curves (of uniformly bounded energy) that converge to a nodal curve u_∞ with components $u_\infty^1, \dots, u_\infty^m$. We define the $\operatorname{ind} u_\infty$ as $\operatorname{ind} u_k$ for a sufficiently large k and consequently have the following relationship between the indices of the components of the limit and the index of the limit:

$$\operatorname{ind} u_\infty = \sum_{r=1}^{m} \operatorname{ind} u_\infty^r + m_r,$$

where m_r is the number of nodal points of u_∞ in the component u_∞^r.[8] Evidently, this constraints the Gromov limit of curves of low index, as then there cannot be many nodal points. Denote by $\overline{\mathcal{M}}_g(A, J)$ the closure of $\mathcal{M}_g(A, J)$ in its Gromov compactification, which we now describe in the cases relevant to us.

[8]This is actually not hard to see. The index of the limit is defined this way because gluing should tell us that this is the correct number of deformations (the virtual dimension of the compactification should be the same as the smooth locus). The nodal curve can now deform as the components deform with constraints on the nodal points. The equality stated follows from using the index formula after relating the Euler characteristic of the glued curve with the Euler characteristic of the components.

Exceptional spheres. An embedded symplectic sphere of self-intersection -1 is called an *exceptional sphere*.

6. * Let $u : (\mathbb{S}^2, j) \to (M, J)$ be a J-holomorphic sphere of self-intersection -1 and write $E := [u]$. Show that it cannot be multiply covered. Describe $\overline{\mathcal{M}}_0(E, J)$. Finally, show that if J can be taken generically, then all J-spheres of self-intersection -1 must be embedded.

Fiber curves An embedded symplectic sphere of self-intersection 0 is called a *fiber curve*. Notice that any J making a fiber curve J-holomorphic produces a local foliation near it (by the foliation property exercise and automatic transversality). The following are the main compactness results for these that we will need. For the following exercises *assume there are no simple holomorphic curves of negative index for that given* J, which is a reasonable assumption due to transversality. Let $(u_k : (\mathbb{S}^2, j_k) \to (M, J))_k$ be a sequence of index 2 somewhere-injective spheres Gromov converging to u_∞.

7. Show that if u_∞ has no nodes, then it is it could potentially be a multiply covered curve, but only by a degree two covering map (as similarly exemplified by the conics degenerating to a doubly covered line in $\mathbb{CP}^2$).

8. * Show that if u_∞ has nodes, it must consist of two embedded sphere components u_∞^1 and u_∞^2 of index 0 (i.e. two exceptional spheres), intersecting transversely once at a point. Show that there can only be finitely many of these degenerations.

   Hint: Show that if it has nodes, there are at most two components (and no ghosts) of index 0, which are not multiply covered. Hence, they are embedded and conclude ruling out the possibility that both components have identical images and that then there is a unique self-intersection.

PART VI: Proofs of Gromov's theorems.

9. * Finish the proof of theorem 1.[9]

10. Sketch a proof of theorem 2.

    Hint: Proceed similarly as before for both fiber classes. Note that the tangent bundles of two symplectic surfaces intersect transversely and positively can still be made J-invariant simultaneously for some J.

11. * Sketch a proof of theorem 3.

12. Explain which part of the argument would fundamentally break down if we wanted to show that a higher genus symplectic surface of self-intersection 0 (so possibly a fiber class) forces a minimal symplectic manifold to be globally fibered by that surface.

Bonus 4. Just using these theorems, what can you say about the minimal fillings of the standard contact $\mathbb{S}^2 \times \mathbb{S}^1$?

Remar. Applications of these theorems. The J-holomorphic "coordinate grid" built on the evenly split $\mathbb{S}^2 \times \mathbb{S}^2$ for any J can be used to compute the group of (compactly supported) symplectomorphisms of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}^2$; the former, up to homotopy, is the rotation of each of the facts and the swap map, and the latter is contractible.

[9] In the argument you show that $\mathcal{M}_0([C], J)$ is a closed genus b surface (for a well chosen J). Implicit in here is that one can orient $\mathcal{M}_0([C], J_C)$ in a natural way, which we have not yet seen in the lectures. Take it as a gray box that will be opened in a couple of weeks.

Remark. Extensions of these theorems If we drop minimality in the first theorem, $M \to \overline{\mathcal{M}}_0(C, J)$ becomes a Lefschetz fibration, which is still a nice theorem. We can also ask what happens when we consider a curve C with self-intersection $k > 0$. For example, in the case of a line in $\mathbb{CP}^2$, we see that the foliation property fails but it still osculates around a point. This is the basic model of a Lefschetz pencil (not quite a fibration). In general, the structure of a symplectic manifold with an embedded symplectic sphere of self-intersection k is that of a Lefschetz pencil osculating at the self-intersection points. These can be shown to be the ruled or rational manifolds (blow ups of $\mathbb{CP}^2$). These are theorems of McDuff.

Remark. Classification of symplectic $\mathbb{S}^2 \times \mathbb{S}^2$'s. We have shown that the presence of two zero-spheres intersecting once transversely forces a symplectic manifold to be equivalent to the symplectically split $\mathbb{S}^2 \times \mathbb{S}^2$. This, however, does not rule out the possibility of a non-split symplectic form, as there is no reason, a priori, to expect it would have such a configuration of spheres. The issue is that we could have cohomologous symplectic forms (they agree on the area of the coordinate spheres) that are not symplectomorphic. This cannot happen, Lalonde and McDuff showed that cohomologous symplectic forms on ruled symplectic 4-manifolds are symplectomorphic. The proof uses Taubes' SW=Gr (or similar results) to produce symplectic fiber classes and concludes via the theorems proved here along with "J-holomorphic inflation".