talk 3 for Atiyah-Singer-Index thm seminar:

charakteristic classes: P_r map from $M(k,\mathbb{C}) \to \mathbb{C}$, r-linear & symmetric, Ad-invariant, $P_r \in I^r(g)$

Chern-Weyl-thm: P an invariant polynomial then (P homogenous polynomial of degree r)

1) • $P(\mathcal{F})$ is closed ($\mathcal{F}$ curvature 2-form) ↳ (curvature transforms on overlaps like $t_{ij}^{-1} \mathcal{F} t_{ij}$ → can work locally but result is independent!)

2) • $\mathcal{F}'$ another curvature 2-form associated to another connection, then $P(\mathcal{F}) - P(\mathcal{F}')$ exact

1): $P(X_1, \dots X_r) = P(g_t^{-1} X_1 g_t, g_t^{-1} X_2 g_t \dots, g_t^{-1} X_r g_t)$ $g_t = e^{Xt}$

$d|_{t=0} \Rightarrow$ $0 = \sum_{i=1}^{r} P(X_1, \dots, [X_i, X], \dots X_r]$ $, X_i, X \in g$

for forms: $P(\omega_1 \dots \omega_r) = \eta_1 \wedge \eta_2 \wedge \wedge \eta_r \, P(X_1, \dots X_r)$

$[\underset{\Omega^2(E,g)}{W_i}, \underset{\Omega^2(E,g)}{W}] = \eta_i \wedge \eta [X_i, X] = \eta_i \wedge \eta (X_i X) + \eta \wedge \eta_i (X X_i)$

↳ $P(\omega_1, \dots [\omega_i, \omega], \dots \omega_r) = \eta \wedge \eta_1 \wedge \dots \eta_r \, P(X_1 \dots [X_i, X], \dots, X_r)$, now $\omega_i = \mathcal{F}_i$

$dP(\mathcal{F}_1 \dots \mathcal{F}_r) = d(\eta_1 \wedge \dots \eta_r) \, P(f_1, \dots, f_r)$

$= \sum_{i=1}^{r} (\eta_1 \wedge d\eta_i \dots \wedge \eta_r) P(f_1, \dots, f_r) = \sum_{i=1}^{r} P(\mathcal{F}_1, \dots, d\mathcal{F}_i, \dots \mathcal{F}_r)$

add a trivial 0: $\sum_{i=1}^{r} P(\mathcal{F}_1 \dots, [\mathcal{F}, A] \dots \mathcal{F}_r) = 0$

$([A_1, A_2] = \eta_1 \wedge \eta_2 [\tilde{A}_1, \tilde{A}_2]$
$= -\eta_2 \wedge \eta_1 [\tilde{A}_1, \tilde{A}_2]$
$= +\eta_2 \wedge \eta_1 [\tilde{A}_2, \tilde{A}_1] = [A_2, A_1]$

↳ $dP(\mathcal{F} \dots \mathcal{F}_r) = \sum_{i=1}^{r} P(\mathcal{F} \dots, \underbrace{[\mathcal{F}_i, A] + d\mathcal{F}_i}_{D\mathcal{F} = 0 \text{ (Cartan identity)}}, \dots \mathcal{F}_r) = 0$ ✓

$| D\mathcal{F} = d\mathcal{F} + [\mathcal{F}, A] = d(dA + [A, A]) + [dA, A] + A \wedge A \wedge A$

10.1.18

2) $A_t = A + t\Theta$, $\Theta = A' - A$

$\mathcal{F}_t = dA_t + A_t \wedge A_t = \mathcal{F} + t\, d\Theta + t(A \wedge \Theta + \Theta \wedge A) + t^2 \Theta^2 = \mathcal{F} + t D\Theta + t^2 \Theta^2$

$P_r(\mathcal{F}) - P_r(\mathcal{F}') = \int_0^1 dt \underbrace{\frac{d}{dt} P_r(\mathcal{F}_t)}_{r P_r(\frac{d\mathcal{F}_t}{dt}, \mathcal{F}_t \dots \overset{r}{\mathcal{F}_t})}$, $\frac{d}{dt} P_r(\mathcal{F}_t) = r P_r(D\Theta + 2t\Theta^2, \mathcal{F}_t, \dots \check{\mathcal{F}}_t)$
$= r P_r(\Theta, \mathcal{F}) + 2rt \, P_r(\Theta^2, \mathcal{F})$

note $D_t \mathcal{F}_t = d\mathcal{F}_t + [A, \mathcal{F}_t] = -[A_t, \mathcal{F}_t] + [A, \mathcal{F}_t] = [A - A_t, \mathcal{F}_t] = t[\mathcal{F}_t, \Theta]$

$(D_t \mathcal{F}_t = 0 = d\mathcal{F}_t + [A_t, \mathcal{F}_t]$

look at $d(P_r(\Theta, \mathcal{F}_t \dots, \overset{r}{\mathcal{F}_t})) = P_r(d\Theta, \mathcal{F}_t \dots \mathcal{F}_t) - (r-1) P_r(\Theta, d\mathcal{F}_t, \mathcal{F}_t \dots)$
$= P_r(d\Theta, \mathcal{F}_t \dots) - (r-1)(P_r(\Theta, d\mathcal{F}_t, \dots \mathcal{F}_t) + P_r(\Theta, [\mathcal{F}_t, A], \mathcal{F}_t)(r-1)$ $+ P_r([\Theta, A], \mathcal{F}_t)$

but we know
$= P_r(D\Theta, \mathcal{F}_t \dots) - (r-1) P_r(\Theta, \underbrace{D\mathcal{F}_t}_{t[\mathcal{F}_t, \Theta]}, \mathcal{F}_t \dots)$

$2P_r(\Theta^2, \mathcal{F}_t, \dots) + (r-1) P_r(\Theta, [\mathcal{F}_t, \Theta], \mathcal{F}_t \dots) = 0$

$\Rightarrow r\, d(P_r(\Theta, \mathcal{F}_t, \dots \mathcal{F}_t)) = \frac{d}{dt} P_r(\mathcal{F}_t)$ ✓

→ in particular for M cpct. $\int_M P_r(\mathcal{F}) \overset{\text{Stokes}}{=} \int_M P_r(\mathcal{F}')$

on $I^*(g) := \bigoplus_{v=0}^{\infty} I^v(g)$, define product: $P_v \in I^v$, $S_t \in I^t$, $(P_v S_t)(X_1, \dots, X_{v+t})$
$= \frac{1}{(v+t)!} \sum_{P_{erm}} P_v(X_{P(1)} \dots X_{P(v)}) S_t(X_{P(v+1)} \dots X_{P(v+t)})$

Coroll: $\rightsquigarrow I^*(g)$ is algebra and get a hom:

3) $\chi_E : I^*(g) \to H^*(M)$ (the Weil-homomorphism)

4) (naturality) $f: N \to M$ diff. map. then $\chi_{f^*E} = f^* \chi_E$ follows from $F_{(f^*A)} = f^* F_A$

Cor 4) charact. classes of trivial bundles are trivial

Chern-classes $E \xrightarrow{\pi} M$ cplx vec. bundle, fibre $\mathbb{C}^k$ structure group $G \subset GL(k, \mathbb{C})$, so A & F have values in g

Def. total chern class $c(F) = \det\left(I + \frac{iF}{2\pi}\right)$, $[c(F)] \in H^*(M)$

expanding det: $c(F) = 1 + c_1(F) + c_2(F) + \dots$, $c_j(F) \in \Omega^{2j}(M)$ is called j-th chern-class, $[c_j(F)] \in H^{2j}(M)$

note that $c_j(F) = 0$ for $j > k$ and for $2j > m = \dim M$ | why $\frac{1}{2\pi}$? $\rightsquigarrow$ want integer integral?

calculating the determinant can be cumbersome $\rightsquigarrow$ diag. F by appropr. matrix $g \in GL(k, \mathbb{C})$

$g^{-1}\left(\frac{iF}{2\pi}\right) g = \mathrm{diag}(x_1 \dots x_k) = X$, then same as if E was direct sum of k line bundles

$c(F) = c(g^{-1} F g) = \det(1 + X) = \prod_{i=1}^{k} (1 + x_i) = \underset{S_0(x_j)}{1} + \underset{S_1(x_j)}{\mathrm{tr}(X)} + \underbrace{\tfrac{1}{2}\left((\mathrm{tr}(X))^2 - \mathrm{tr}(X^2)\right)}_{S_2(x_j)} + \dots + \det X$ are the elem. symmetric polynomials

since these S_i generate the polynomial ring, we see that the chern classes c_k correspond to the elem. symm. polynom.

Properties of chern classes (in a way paradigm. for charact. classes)

- $c(f^*E) = f^* c(E)$, $f: N \to M \xleftarrow{\pi} E$ (naturality) $\Leftarrow$ clear from 4), but can also check directly
- $c(E \oplus F) = c(E) \wedge c(F)$ (Whitney sum) $\Leftarrow$ assume $F_{E \oplus F}$ is block diagonal

For the A.S.I theorem, need a different characteristic class

Chern-character

Def: total chern character $ch(F) = \mathrm{tr}\left(\exp\left(\frac{iF}{2\pi}\right)\right) = \sum_{j=1} \underbrace{\frac{1}{j!} \mathrm{tr}\left(\frac{iF}{2\pi}\right)^j}_{ch_j(F)} \in H^*(M)$

$ch_j(F)$ the j-th chern character, $\in H^{2j}(M)$

note, again $ch_j(F) = 0$ for $2j > m$ $\rightsquigarrow$ $ch(F)$ is finite polynomial

diagonalise ... $\to$ $ch(F) = \sum_{j=1}^{k} \exp(x_j) = k + S_1(x_j) + \frac{1}{2}\left(S_1(x_j)^2 - 2S_2(x_j)\right) + \dots$

so $ch_0(F) = k$, $ch_1(F) = c_1(F)$, $ch_2(F) = \frac{1}{2}\left(c_1(F)^2 - 2c_2(F)\right)$, can be expressed in terms of chern classes (as expected)

Properties of chern character: • $ch(f^*E) = f^*(ch(E))$

• $ch(E\otimes F) = ch(E)\wedge ch(F) \leftarrow F_{E\otimes F} = F_E \otimes I + I \otimes F_F$

• $ch(E\oplus F) = ch(E) + ch(F) \leftarrow$ curvature blockdiagonal

holomorphic functions can be used to define further charact. classes.

Def: f holom. near 0, then the chern-f-genus is defined by $\Pi_f(X) = \det(f(\frac{i}{2\pi}X)) = \prod f(x_j)$

Chern-class (total): $f = 1+z$

Pontrjagin-classes $E \xrightarrow{\pi} M$ real vec. b., $\dim_{\mathbb{R}} E = k$, endowing E with fibre metric, may reduce structure group to $O(k) \subset GL(k,\mathbb{R})$, but then F is in $\mathfrak{o}(k)$, so is skew symmetric.

↳ not diag. by an element of a subgroup of $GL(k,\mathbb{R})$, but get sth like

$F \rightarrow X = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix} \xrightarrow{\text{use } g\in GL(k,\mathbb{C})} \begin{pmatrix} i\lambda_1 & & & & \\ & -i\lambda_1 & & & \\ & & i\lambda_2 & & \\ & & & -i\lambda_2 & \\ & & & & \ddots \end{pmatrix}$

Def: total Pontrjagin class is the total chern class of complexified vec. bundle $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$

$p(F) = \det\left(1 + \frac{F}{2\pi}\right) = \det\left(1 + \frac{F^t}{2\pi}\right) = \det\left(1 - \frac{F}{2\pi}\right) \rightsquigarrow$ only even forms (in F) remain

$= 1 + p_1(F) + p_2(F) + \ldots \qquad p_j(F) \in H^{4j}(M,\mathbb{R})$ with the obvious ones vanishing

diagonalising: $p(F) = \prod^{[k/2]} (1 + x_i^2) \rightsquigarrow p_j(F) = \sum_{i_1<i_2<\ldots<i_j}^{[k/2]} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_j}^2$

can show $p_j(E) = (-1)^j c_{2j}(E^{\mathbb{C}})$

holomorphic fct. for the real vec. bundle case:

Def: g - hol. near $z=0$, $g(0)=1$, f the branch of $z \mapsto (g(z^2))^{\frac{1}{2}}$ with $f(0)=1$
the pontrjagin-g-genus of a real vec. bundle is the Chern f-genus of its complexif.

Lemma: $\underbrace{\Pi_g}_{\text{Pontrjagin g-genus}}(X) = \prod g(y_i)$ ↲ formal variable y_i corresponds to j-th pontrjagin class

two important examples:

• Hirzebruch L-polynomial : $L(x) = \prod^{k} \frac{x_j}{\tanh x_j}$ ← pontrjagin-$\frac{\sqrt{z}}{\tanh(\sqrt{z})}$-genus
is for Hirzebruch signature thm $= \prod^{k}\left(1 + \sum_{n\geq 1} (-1)^{n-1} \frac{2^{2n}}{(2n)!} B_n x_j^{2n}\right)$

• $\hat{A}(F) = \prod_{j=1}^{k} \frac{x_j/2}{\sinh(x_j/2)}$ (pontrjagin-$\frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$-genus)

Now calculate some charact. classes for spin bundles:

Let M be a spin mfld of even dim. $2m$, Δ the associated spin bundle

Proposition (4.2.3, Roe): $ch(\Delta) = 2^m \Pi_g(TM)$, $g(z) = \cosh(\frac{1}{2}\sqrt{z})$

relative chern character

Def: relative trace of a Clifford module Endomorphism F of a repr. $W \overset{!}{=} \Delta \otimes V$ ← some vec. space
can always be written that way

, $tr^{W/\Delta}(F) = tr(\tilde{F})$, where $\tilde{F}$ is the identification of F under $\mathrm{End}_{Cl(E)\otimes\mathbb{C}}(W)$
$\approx \mathrm{End}_{\mathbb{C}\text{-vec}}(V)$

Def: (relative chern character) Let S be a Clifford bundle over M, then,
... is $ch(S/\Delta) := tr^{S/\Delta}(\exp(\frac{i}{2\pi}F^S))$

analog. to prop (4.2.3,) we get for gen. Cliff. bdl S: $ch(S) = 2^m \Pi_g(TM) \wedge ch(S/\Delta)$

Remark: F^S for M spin vanishes → $ch(S/\Delta)$ trivial!

Now we have all the ingredients to write down the Atiyah-Singer Index thm:

[ATIYAH-SINGER] M cpt, even dim. oriented mfld, S graded Clifford bundle w/ Dirac op. D,
index $(D) = \int_M \hat{A}(TM) \wedge ch(S/\Delta)$

Todd class & Euler class

Def: Todd class $\quad Td(F) = \prod_j \frac{x_j}{1-e^{-x_j}}$

Euler class $\quad e(F) = Pf(\frac{F}{2\pi})$
(M even dim. and orientable)