Getzler Calculus

Filtered algebras and symbols

Def: A an algebra over $\mathbb{C}$.

(i) Grading of A is $A = \bigoplus A^m$ with $A^m \cdot A^{m'} \subseteq A^{m+m'}$

(ii) Filtration of A is $\dots \subseteq A_m \subseteq A_{m+1} \subseteq \dots$ s.t. $A_m \cdot A_{m'} \subseteq A_{m+m'}$.

Examples: · Algebra $D(M)$ of diff. op. $C^\infty(M,\mathbb{C}) \to C^\infty(M,\mathbb{C})$. $D_m(M)$ = diff. op of order $\leq m$.
· Clifford algebra $Cl(V)$: $Cl_m(V)$ = span of products with $\leq m$ elements of V.

Def: Let A filtered algebra. Associated graded algebra is $\mathcal{G}(A) = \bigoplus A_m / A_{m-1}$.

Def: Let A filtered, $\mathcal{G}$ graded. Symbol map $\sigma_\bullet : A \to \mathcal{G}$ is family $\sigma_m : A_m \to \mathcal{G}^m$ s.t.

(i) $a \in A_{m-1} \Rightarrow \sigma_m(a) = 0$

(ii) $a \in A_m,\ a' \in A_{m'} \Rightarrow \sigma_m(a)\sigma_{m'}(a') = \sigma_{m+m'}(aa')$. (hom-like)

For A filtered, we always have the universal symbol map $\sigma_\bullet : A \to \mathcal{G}(A)$, $\sigma_m = \pi : A_m \to A_m/A_{m-1}$.

Example 1: $A = Cl(V) \Rightarrow \mathcal{G}(A) = \Lambda^* V$. Then $\sigma_\bullet : Cl(V) \to \Lambda^* V$ is top degree of iso $Cl(V) \cong \Lambda^* V$ as $\mathbb{C}$-vs.

Example 2: Let $A = D(M)$. Let $C(V)$ = const coeff diff op on $C^\infty(V,\mathbb{C})$. Define symbol $\sigma_\bullet : D(M) \to C^\infty(C(TM))$ by locally $\sum_{|\alpha| \leq m} c_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} \rightsquigarrow \sum_{|\alpha| = m} c_\alpha(x_0) \frac{\partial^\alpha}{\partial x^\alpha}$. $\rightsquigarrow$ same as before

Remark: $D(M)$ generated by $C^\infty(M)$ (deg 0) and $\mathfrak{X}(M)$ (deg 1). Therefore $\sigma_\bullet$ def by image on these generators: $\sigma_0(f) = f$ for $f \in C^\infty(M)$, $\sigma_1(X) = X$ ($= \partial_{X_p}$ on T_pM).

Getzler symbols

Goal: Study diff op alg $D(S)$ for Clifford bndl S. Recall $End(S) = Cl(TM) \otimes End_{Cl}(S)$. Hence $D(S)$ generated by $Cl(TM)$, $End_{Cl}(S)$ and covariant derivatives

Def: The Getzler filtration on $D(S)$ is def. on generators as follows:

(i) $End_{Cl}(S)$: deg 0

(ii) $c(X)$ for $X \in \mathfrak{X}(M)$: deg 1

(iii) ∇_X for $X \in \mathfrak{X}(M)$: deg 1

Def: V a $\mathbb{R}$-vs. Then $\mathcal{P}(V)$ = diff. op on $C^\infty(V,\mathbb{C})$ with polynomial coeff. Graded by $\deg\left(x^\alpha \frac{\partial^\beta}{\partial x^\beta}\right) = |\beta| - |\alpha|$.

Example: Riemann curvature on M is 2-form with values in $End(TM)$. Let $X \in \mathfrak{X}(M)$, $p \in M$. Get linear map $v \mapsto (R_p X_p, v) : T_pM \to \Lambda^2 T_p^*(M)$. Identify T_p^*M with T_pM $\rightsquigarrow$ $T_pM \to \Lambda^2 T_pM$.
$\rightsquigarrow$ Denote by $(RX, \cdot) \in \mathcal{P}(TM) \otimes \Lambda^* TM$.

Remark: $\mathcal{G}_1, \mathcal{G}_2$ graded $\rightsquigarrow$ canonical grading on $\mathcal{G}_1 \otimes \mathcal{G}_2$: $(\mathcal{G}_1 \otimes \mathcal{G}_2)^m = \sum_{k+\ell=m} \mathcal{G}_1^k \otimes \mathcal{G}_2^\ell$

Prop: $\exists!$ symbol map

$$\sigma_\cdot : \mathcal{D}(S) \longrightarrow C^\infty(\mathcal{P}(TM) \otimes \Lambda^* TM \otimes \mathrm{End}_{c\ell}(S))$$

satisfying

(i) $\sigma_0(\mathcal{F}) = 1 \otimes 1 \otimes \mathcal{F}$ for $\mathcal{F} \in \mathrm{End}_{c\ell}(S)$

(ii) $\sigma_1(c(X)) = 1 \otimes X \otimes 1$ for $X \in \mathfrak{X}(M)$

(iii) $\sigma_1(\nabla_X) = \partial_X + \frac{1}{4}(RX, \cdot)$ for $X \in \mathfrak{X}(M)$.

Example 1: In $\mathcal{D}(S)$ we have

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = K(X,Y) = R^S(X,Y) + \mathcal{F}^S(X,Y)$$

$$R^S(X,Y) = \frac{1}{4}\sum_{k,\ell} c(e_k)c(e_\ell)(R(X,Y)e_k, e_\ell) \in \mathrm{End}(S) \qquad (e_i \text{ basis of } TM \text{ locally})$$

$$\mathcal{F}^S(X,Y) \in \mathrm{End}_{c\ell}(S)$$

Compute Getzler symbol of both sides:

1) LHS: $\nabla_{[X,Y]}$ has deg 1 $\rightsquigarrow$ ignore.

Let e_i basis of T_pM with coord x^i. Then

$$\sigma_1(\nabla_i) = \frac{\partial}{\partial x^i} + \frac{1}{4}\underbrace{(Re_i, \cdot)}_{\left[v = \sum_j v^j e_j \mapsto \sum_j v^j (Re_i, e_j) = \frac{1}{2}\sum_{j,k,\ell} v^j (R(e_k, e_\ell)e_i, e_j)\, e_k \wedge e_\ell\right]} = \frac{\partial}{\partial x^i} + \frac{1}{8}\sum_{j,k,\ell}(R(e_k, e_\ell)e_i, e_j)x^j\, e_k \wedge e_\ell = \frac{\partial}{\partial x^i} + \frac{1}{8}\sum (R(e_i, e_j)e_k, e_\ell)x^j e_k \wedge e_\ell$$

In $[\sigma_1(\nabla_i), \sigma_1(\nabla_j)]$, only cross-terms remain:

$$\sigma_2(\nabla_i \nabla_j - \nabla_j \nabla_i) = \frac{1}{4}\sum_{k,\ell}(R(e_i, e_j)e_k, e_\ell)\, e_k \wedge e_\ell$$

2) RHS: $\sigma_2(R^S(e_i, e_j)) = \text{LHS}$.

$\sigma_2(\mathcal{F}^S) = 0$.

Example 2: Dirac operator D: deg 2 and $\sigma_2(D) = d_{TM}$ (exterior derivative on TM):

Let e_i orthonormal frame $\rightsquigarrow D = \sum c(e_i)\nabla_i$. Then

$$\sigma_2(D) = \sum_i e_i \frac{\partial}{\partial x^i} - \frac{1}{8}\underbrace{\sum_{i,j,k,\ell}(R(e_i, e_j)e_k, e_\ell)x^j e_i \wedge e_k \wedge e_\ell}_{= \sum (R(e_k, e_\ell)e_i, e_j)x^j e_i \wedge e_k \wedge e_\ell = 0 \text{ by Bianchi identity } (R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0)} = \sum_i e_i \frac{\partial}{\partial x^i}$$

In particular $\sigma_4(D^2) = 0$. In fact D^2 has Getzler order 2:

Prop: D^2 has Getzler order 2. Symbol rel. orthonorm basis of T_pM is

$$\sigma_2(D^2) = -\sum_i\left(\frac{\partial}{\partial x^i} + \frac{1}{4}\sum R_{ij}x^j\right)^2 + \mathcal{F}^S \qquad (R_{ij} = (Re_i, e_j))$$

Proof: Weitzenbock formula:

$$D^2 = \nabla^*\nabla + \frac{1}{4}\kappa + \underline{\mathcal{F}}^S$$

$$\nabla^*\nabla = \sum_i -\nabla_i^2$$

$$\underline{\mathcal{F}}^S = \sum_{k,\ell} c(e_k)c(e_\ell)\mathcal{F}^S(e_k, e_\ell) \qquad \text{(Clifford contraction)}$$

Then $\sigma_2(\nabla^*\nabla) = $ first term, $\sigma_2(\kappa) = 0$, $\sigma_2(\underline{\mathcal{F}}^S) = \mathcal{F}^S$. $\square$

Getzler symbol of smoothing operators

Goal: Define and analyse Getzler symbol of heat kernel.
$\rightsquigarrow$ Need Getzler symbol for smoothing operators (which are not differential op.) on S.

Def: V a $\mathbb{C}$-vs $\rightsquigarrow$ $\mathbb{C}[[V]] := \prod_{i=0}^{\infty} \otimes^i V$ the ring of formal power series in V.
Grading: $\deg(x^\alpha) = -|\alpha|$. $\rightsquigarrow$ $\mathbb{C}[[V]]$ is graded $P(V)$-module.

Let $s \in \Gamma(S \boxtimes S^*)$, i.e. $s : M \times M \ni (p,q) \mapsto s_q(p) \in \mathrm{Hom}(S_q, S_p)$. Recall assoc. smoothing op: $\Gamma(S) \ni \sigma \mapsto [p \mapsto \int_M s(p,q)\sigma(q)\,dq]$
Fix $q \in M$, choose geodesic coord x^i around q. Then $s_q(x) \sim \sum_\alpha s_\alpha x^\alpha$ with s_α parallel along geodesics starting in q (Taylor series). $\rightsquigarrow$ s_α determined by $s_\alpha(0) \in \mathrm{End}(S_q)$ $\rightsquigarrow$ Taylor series $\in \mathbb{C}[[T_qM]] \otimes \mathrm{End}\,S_q$
$\rightsquigarrow$ obtain section $\Sigma(s) \in \mathbb{C}[[TM]] \otimes \mathrm{End}(S)$.

Def: Filtration on $\Gamma(S \boxtimes S^*)$: s has $\deg \le m$ if $\Sigma(s)_p \in \mathbb{C}[[T_pM]] \otimes \mathrm{End}(S_p)$ has $\deg \le m$ $\forall p \in M$.
Symbol map. (Getzler symbol)

$$\sigma_\cdot : \Gamma(S \boxtimes S^*) \to C^\infty(\mathbb{C}[[TM]] \otimes \Lambda^* TM \otimes \mathrm{End}_{\mathcal{C}\ell}(S))$$
$$= \left(\mathrm{End}(S_q) = \mathcal{C}\ell(T_qM) \otimes \mathrm{End}_{\mathcal{C}\ell}(S_q) \to \Lambda^* T_qM \otimes \mathrm{End}_{\mathcal{C}\ell}(S_q)\right) \circ \left(\Sigma : \Gamma(S \boxtimes S^*) \to \mathbb{C}[[TM]] \otimes \mathrm{End}(S)\right)$$

Remark: σ not hom-like wrt comp of smoothing op.

Prop: Let T be one of the generators of $D(S)$ i.e. $T \in \mathrm{End}_{\mathcal{C}\ell}(S)$, $T = c(X)$ or $T = \nabla_X$ $(X \in \mathfrak{X}(M))$, $\deg(T) = m \in \{0,1\}$.
Let Q smoothing op on $C^\infty(S)$, $\deg Q \le k$. Then TQ smoothing op of $\deg \le m+k$ and

$$\sigma_{m+k}(TQ) = \sigma_m(T)\sigma_k(Q)$$
$\quad$ composition = $P(TM)$-module structure of $\mathbb{C}[[TM]]$.

Cor: Getzler symbol well-defined on $D(S)$ and

$$\sigma_{m+k}(TQ) = \sigma_m(T)\sigma_k(Q) \qquad \forall T \in D(S),\ Q \text{ smoothing}$$

Proof: Wts $\sigma_m(T)$ indep of repr. Let $\tilde{T}$ one repr of T. By Prop, $\sigma_{m+k}(TQ) = \sigma_m(\tilde{T})\sigma_k(Q)$. Q arbitrary $\rightsquigarrow$
$\rightsquigarrow$ $\sigma_m(\tilde{T})$ determined by T. $\square$

Proof of Prop: Let s kernel of Q, fix $q \in M$, geodesic coord x^i around q.

Case 1: $T = F \in \mathrm{End}_{\mathcal{C}\ell}(S)$.
$\rightarrow$ If F parallel along geodesics from q $\Rightarrow$ $\Sigma(Fs) = F \cdot \Sigma(s)$ ✓
In general: Let F_0 const term in Taylor expansion $\Rightarrow$ $\sigma_0(F - F_0) = 0$ by deg $\Rightarrow$
$$\Rightarrow \sigma_k(Fs) = \sigma_k(F_0 s) = \sigma_0(F_0)\sigma_k(s) = \sigma_0(F)\sigma_k(s). \checkmark$$

Case 2: $T = c(X)$, $X \in \mathfrak{X}(M)$. $\rightsquigarrow$ Analogous.

Case 3: $T = \nabla_X$. Let ∂_i VF associated to x^i. Wlog $X = \partial_i$. Let $Y = \sum_j x^j \partial_j$.
$\hookrightarrow$ Suppose s parallel along geodesics. Then $\nabla_Y s = 0$. Write $\nabla_X s \sim \sum t_\alpha x^\alpha$. We have

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s = K(X,Y)s, \quad \nabla_Y s = 0, \quad [X,Y] = X, \quad Y x^\alpha = |\alpha| x^\alpha$$
$$\Rightarrow -\sum_\alpha (|\alpha|+1) t_\alpha x^\alpha \sim K(X,Y)s = \sum_j K_{ij} x^j s \qquad (K_{ij} = K(\partial_i, \partial_j),\ X = \partial_i,\ Y = \sum x^j \partial_j.)$$

Now use $K = R^S + F^S$, so that

$$\nabla_X s = -\frac{1}{2}\sum_j x^j R^S(\partial_i, \partial_j)s + \text{lower order terms} \Rightarrow \sigma_{k+1}(\nabla_X s) = \frac{1}{4}\sum_j R_{ij} x^j \wedge \sigma_k(s) = \sigma_1(\nabla_X)\sigma_k(s)$$