1

The Spin Group

Today we'll talk about Spin-Groups and related structures associated to some Manifolds, called Spinstructures. Let me try to give some reasons why you should care about Spin groups/structures:

- They come up naturally in the formulation of the Atiyah-Singer Indexthm.
- Given a compact Manifold, one may integrate the $\hat{A}$-Genus over M: $\hat{A}(M) := \int_M \hat{A}(TM) \in \mathbb{Q}$ this can be shown

Question: Is $\hat{A}(M) \in \mathbb{Z}$?

"Answer": If M admits a spinstructure, then it is!

- If you are interested in understanding smooth structures of top. 4-Manifolds, there is a well-established tool called "Seiberg-Witten equations", which use the notion of Spin$^{\mathbb{C}}$-structures

⋮

Recall: Clifford Algebras

Def: Let (V, f) be a $\mathbb{R}(\mathbb{C})$ quadratic vectorspace ~~vectorspace with a symmetric bilinear form~~ f. A Clifford algebra for (V,f) is an associative algebra $Cl(V)$ with unit 1 equipped with a map $\beta: V \to Cl(V)$ s.t $\beta(v)^2 = -f(v,v)\cdot 1$ satisfying the following universal property:

$\forall$ associative algebras with unit 1 A' which are equipped with a map $\alpha: V \to A'$ s.t $\alpha(v)^2 = -f(v,v)1$ there exist a unique algebra hom. $\gamma: Cl(V) \to A'$ s.t $\gamma \circ \beta = \alpha$

2

The Cliffordalgebra as a Superalgebra

Def: A $\mathbb{K}$ - Vectorspace V is called Supervectorspace if there is a decomposition $V = V_0 \oplus V_1$, ie. V is a $\mathbb{Z}_2$-graded V.S.

For a homogenious element $v \in V$: denote the degree of ~~an element~~ v by $\deg(v) = \begin{cases} 0 & \text{if } v \in V_0 \\ 1 & \text{if } v \in V_1 \end{cases}$

Examples: • $\mathbb{C}$ as a $\mathbb{R}$ - VS, $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$

• The exterior algebra of a vectorspace V,

$$\Lambda^* V = \Big(\bigoplus_{j \geq 0} \Lambda^{2j} V\Big) \oplus \Big(\bigoplus_{j \geq 0} \Lambda^{2j+1} V\Big)$$

Def: A $\mathbb{K}$- superalgebra is a $\mathbb{K}$- Supervectorspace $A = A_0 \oplus A_1$ together with a multiplication $\cdot : A \times A \to A$ s.t $A_i \cdot A_j \subset A_{i+j \,(\mathrm{mod}\, 2)}$ (and which is a ~~associative~~ Ring with respect to addition in A.

To see that the Cliffordalgebra is in fact a superalgebra, look at the Tensoralgebra of V first:

$T(V) = \sum_{r=0} \otimes^r V$, This has an obvious $\mathbb{Z}$-grading.

The Clifford algebra of V is given by

$\mathcal{Cl}(V) \simeq T(V)/I_f$, where I_f is the Ideal

$I_f := \mathrm{span}\{A \otimes (x \otimes x + f(x,x) \cdot \mathbb{1}) \otimes B \mid A, B \in T(V)\}$

We see that the $\mathbb{Z}$-grading of $T(V)$ does not descent to $\mathcal{Cl}(V)$, however, the $\mathbb{Z}_2$-grading does!

More explicitly:

$\mathcal{Cl}(V)_0 = \mathrm{span}\{e_{j_1} \dots e_{j_{2k}} \mid \{e_{j_i}\} \text{ a chosen basis of } V, k \in \mathbb{N}\}$

$\mathcal{Cl}(V)_1 = \mathrm{span}\{e_{j_1} \dots e_{j_{2k+1}} \mid \quad\}$

For later purposes, define the supercommutator and the supercenter:

Def: For A a superalgebra, define the supercommutator for hom. elements

$[x,y]_s := xy - (-1)^{\deg(x)\deg(y)} \cdot yx$, $x, y \in A$ hom. ($\substack{y \\ x} \in A_i$)

Define the supercenter of A by $Z_s(A) := \{y \in A \mid [x,y]_s = 0 \; \forall x \in A\}$

3 Lemma: If (V, f) is a real (~~complex~~) quadratic vectorspace ~~V.S. with a symmetric bilinear form f~~, then $Z_s(Cl(V)) = \mathbb{R}$ & $Z_s(Cl(V) \otimes \mathbb{C}) = \mathbb{C}$

Pf: (Roe chapter 4, Lemma 4.3)

Let $e_1 \dots e_k$ be an ONB for V, write $x \in Cl(V)$ as

$x = a + e_1 b$, s.t. a and b don't contain e_1's

(can be achieved by commuting elements in b and using $e_1^2 = -1$)

Assume x homogenious, $\deg(x) = \deg(a) = \underbrace{\deg(e_1)}_{=1} + \deg(b)$

Now $x e_1 = a e_1 + e_1 b e_1$ and $e_1 x = e_1 a - b$

$$= (-1)^{\deg(x)}(e_1 a - e_1^2 b)$$

$$= (-1)^{\deg(x)}(e_1 a + b)$$

By assumption, $x \in Z_s(Cl(V))$, so $[x, e_1]_s = 0$

but $[x, e_1]_s = x e_1 - (-1)^{\deg(x)\underbrace{\deg(e_1)}_{=1}} e_1 x = (-1)^{\deg(x)}(e_1 a + b - e_1 a + b)$

$$= (-1)^{\deg(x)} 2b$$

$\hookrightarrow b = 0 \Rightarrow x$ does not contain e_1

$\hookrightarrow x$ does not contain any e_i's $\Rightarrow x$ is a scalar. ∎

For the representation theory of $Cl(V)$ and the spingroups, the following definition comes in handy:

Def: The volume element in $Cl(V)$, $\omega_V (= \omega)$ is the product $\omega_V = e_1 \dots e_k$ where $\{e_1 \dots e_k\}$ is a chosen positively oriented ONB for V

A quick calculation gives: $\omega^2 = (-1)^{\frac{k(k+1)}{2}}$, $\omega v = (-1)^{k-1} v \omega \;\; \forall v \in V$

$\omega^2 = (e_1 \dots e_k)(e_1 \dots e_k) = (-1)^{k-1}(e_1 e_1)(e_2 e_3 \dots e_k)(e_2 e_3 \dots e_k)$

(to commute the 2^{nd} e_1 next to the 1^{st}, have to commute past $(k-1)$ basis-elements $e_2 e_3 \dots e_k \to (-1)^{k-1}$)

$$= (-1)^{(k-1)+(k-2)+(k-3)+\dots+1} e_1^2 e_2^2 e_3^2 \dots e_k^2 = (-1)^{\sum_{i=1}^{k} i} = (-1)^{\frac{k(k+1)}{2}}$$

$\omega e_i = (-1)^{k-1}$

must commute past k elements $e_1 \dots e_k$, but since e_i is in there, only $(k-1)$ minus signs..

Pin and Spin

Let $\mathcal{Cl}(n)$ be the Clifford algebra of $(\mathbb{R}^n, \langle x,y\rangle_{eucl.})$

Def: $Pin(n) := \{ x_1 \dots x_\ell \mid x_j \in \mathbb{R}^n, \langle x_j, x_j\rangle = 1 \}$ | Examples: $Pin(1) \simeq \mathbb{Z}_4$, $Spin(1) \simeq \mathbb{Z}_2$

$Spin(n) := \{ x_1 \dots x_{2\ell} \mid x_j \in \mathbb{R}^n, \langle x_j, x_j\rangle = 1 \} = Pin(n) \cap \mathcal{Cl}_0(n)$

It is straightforward to show that these are groups. However, from the above definition one may not see why they are important. Let's define a map

$$\rho : Pin(n) \to Aut(\mathcal{Cl}(n)), \quad \rho(y)x = (-1)^{deg(y)} y x y^{-1}$$

How does this act on $\mathbb{R}^n \subset \mathcal{Cl}(n)$? For v a unit vector $(v \in Pin(n))$, $x \in \mathbb{R}^n$

$$\rho(v)x = -v x \underbrace{v^{-1}}_{=-v} = \underbrace{v x}_{} v = -vvx - 2v\langle x,v\rangle$$
$$-vx - 2\langle x,v\rangle \cdot \mathbb{1} \quad (\text{Clifford mult.})$$
$$= x - 2\langle x, v\rangle v$$

A.: It acts by reflections! (reflection of x in the hyperplane perpendicular to v)

ρ is in fact a rep. of $Pin(n)$, since we have

$$\rho(y_1 y_2)x = (-1)^{deg(y_1 y_2)} y_1 y_2 x (y_1 y_2)^{-1} = (-1)^{deg(y_1)+deg(y_2)} y_1 (y_2 x y_2^{-1}) y_1^{-1}$$
$$= \rho(y_1)(\rho(y_2)x)$$

We now use a classical lemma about the orthogonal groups $O(n)$

Lemma: Any $A \in O(n)$ may be written as a product of reflections, if $A \in SO(n)$ then it may be written as ~~a even num~~ a product of an even number of reflections. (Proof: see Hein: Klassische Gruppen)

Thus, since ~~for~~ $u = x_1 \dots x_\ell \in Pin(n)$ $\rho(u) = \rho(x_1) \circ \dots \circ \rho(x_\ell)$

$$= S_{x_1} \circ \dots \circ S_{x_\ell} \in O(n)$$

reflection at x_1

we obtain a surjective map

$\rho : Pin(n) \to O(n)$ and similarly $\rho : Spin(n) \to SO(n)$

Proposition: There is an exact sequence

$$0 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\rho} SO(n) \to 0$$
$$(0 \to \mathbb{Z}_2 \to Pin(n) \to O(n) \to 0)$$

Pf: We only need to show that $\ker \varrho = \{\pm 1\}$

Let $x \in \ker \varrho$, so $\varrho(x)v = -\cancel{\ } x v x^{-1} = v \quad (\forall v \in \mathbb{R}^n)$

$$\Longleftrightarrow \quad xv = vx$$
$$\Longleftrightarrow \quad xa = ax \quad \forall a \in \mathcal{Cl}(n)$$

$\Rightarrow \quad x \in Z_s(\mathcal{Cl}(V)) = \mathbb{R}$

Da $x \in Spin(n)$ muss $x^2 = 1 \Rightarrow \underline{x = \pm 1}$ ∎

Now use a Lemma: If $\lambda : G \to H$ is a contin., surjective grouphom of top. groups G & H and $\ker(\lambda)$ is finite, then λ is a $\# \ker(\lambda)$-fold covering.

Thus we are in a better position to understand $Spin(n)$, namely it is a double covering group of $SO(n)$. (Similary: $Pin(n)$ is a double covering of $O(n)$)

In particular, $Spin(n)$ is a compact Liegroup. The following proposition displays $Spin(k)$ as the universal cover of $SO(k)$ for $k \geq 3$:

Prop: For $k \geq 2$, $Spin(k)$ is connected, for $k \geq 3$ it is simply connected.

Pf: Have exact sequence $0 \to \mathbb{Z}_2 \to Spin(k) \to SO(k) \to 0$

$\overset{\pi_i}{\Rightarrow}$ get exact sequence

$$\underset{\overset{\parallel}{0}}{\pi_1(\mathbb{Z}_2)} \to \underset{\overset{\parallel\, k\geq 3}{\mathbb{Z}_2}}{\pi_1(Spin(k))} \to \pi_1(SO(k)) \to \underset{\overset{\parallel}{\mathbb{Z}_2}}{\pi_0(\mathbb{Z}_2)} \to \pi_0\, Spin(k) \to \underset{\overset{\parallel}{0}}{\pi_0(SO(k))}$$

Thus it suffices to show the map $\pi_0 \mathbb{Z}_2 \to \pi_0\, Spin(k)$ is trivial, which is true if we find a path between $+1$ and -1 in $Spin(k)$.

This can be done by $t \mapsto \cos t + e_1 e_2 \sin t$ provided that $k \geq 2$. ∎

Before getting to the Liealgebra of $Spin(k)$, let us look again at the homotopy groups of $O(n)$: $(n \geq 3)$

	$O(n)$	$SO(n)$	$Spin(n)$	... $String(n)$
π_0	$\mathbb{Z}_2$	0	0	0
π_1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
π_2	0	0	0	0
π_3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	0
$\vdots$				

In the table we see, that moving to the right, we "kill" the homotopy-groups of $O(n)$ from below. To make you curious and suggesting that this can be made rigorous (Postnikov / Whitehead-towers) I added a fourth entry "String(n)", which kills the 3rd homotopy group (but which can therefore not be a Liegroup (finite))

Lemma: The Lie algebra of Spin(k) (and of Pin(k)) is given by $(\Lambda^2\mathbb{R}^k, [x,y] = \text{commutator})$

Pf: We first show that $\Lambda^2\mathbb{R}^k = \operatorname{span}\{e_i \cdot e_j \mid 1 \leq i < j \leq k,\ e_i \in \mathbb{R}^k\}$ is a Lie algebra:

$$[e_ie_j, e_ke_\ell] = e_ie_je_ke_\ell - e_ke_\ell e_ie_j$$
$$= \begin{cases} 0 & \text{for } i \neq k,\ j \neq \ell \\ -2e_je_\ell & \text{for } i = k,\ j \neq \ell \\ 2e_ie_k & \text{for } i \neq k,\ j = \ell \end{cases}$$

$\hookrightarrow (\Lambda^2\mathbb{R}^k, [\,,\,])$ is a Lie algebra

Now we look at $T_1 \operatorname{Spin}(k)$, ~~define~~ a curve $\gamma_{i,j}$ through $1 \in \operatorname{Spin}(k)$

$1 \leq i < j \leq k \qquad \gamma_{i,j}(t) := -(\cos t\, e_i + \sin t\, e_j)\cdot(\cos t\, e_i - \sin t\, e_j) \in \operatorname{Spin}(k)$

- $\gamma_{i,j}(0) = 1$
- $\gamma'_{i,j}(0) = -(e_je_i + e_i(-e_j)) = 2e_ie_j \in \Lambda^2\mathbb{R}^k \subset T_1\operatorname{Spin}(k)$

Since these span $\Lambda^2\mathbb{R}^k$

and $\dim(T_1\operatorname{Spin}(k)) = \dim(SO(k))$

and $\dim(SO(k)) = \frac{k(k-1)}{2} = \binom{k}{2} = \dim(\Lambda^2\mathbb{R}^k)$

we get the desired isomorphism.

Representation theory of $Cl(n)$ and $\operatorname{Spin}(n)$

Given a rep. of $SO(n)$, $\alpha: SO(n) \to GL(\mathbb{R})$, we clearly get a rep. of $\operatorname{Spin}(n)$ by $\alpha \circ \varrho$. In the following we will discuss representations which don't come from a $SO(n)$ rep..

We'll be interested in reps of $Cl(n) \otimes \mathbb{C} =: Cl^{\mathbb{C}}(n)$, since these Clifford algebras have a simple classification:

Prop: $Cl^{\mathbb{C}}(k) \underset{\phi_k}{\overset{\text{algebra isom.}}{\simeq}} \begin{cases} \cancel{\mathbb{C}(2^{N})} = \cancel{\mathbb{C}(2^{N})}\ M(2^m, \mathbb{C}) & k = 2m \\ \cancel{\mathbb{C}(2^N) \oplus \mathbb{C}(2^N)}\ M(2^m, \mathbb{C}) \oplus M(2^m, \mathbb{C}) & k = 2m+1 \end{cases}$

Pf: By giving the isomorphism ϕ_k directly ...

With that proposition we may define two reps of $Cl^{\mathbb{C}}(n)$ over $\mathbb{C}^{2^{\lceil \frac{n}{2} \rceil}}$

$$\kappa_n^{(i)} := \begin{cases} \phi_n & , n = 2m \\ pr_i \circ \phi_n & , n = 2m+1 \end{cases}$$

Def: We have two representations $\kappa_n^{(i)} : Cl^{\mathbb{C}}(n) \to GL(\mathbb{C}^{2^{\lceil \frac{n}{2} \rceil}})$ which are inequivalent. Restricting them to $Spin(n)$ we obtain the spin-representation. We write $\Delta_n = \mathbb{C}^{2^{\lceil \frac{n}{2} \rceil}}$ for the representation space and call elements $\varphi \in \Delta_n$ (complex) spinors. Δ_n becomes via $K_n^1|_{Spin(n)}$ a $Spin(n)$-module.

Remark: The spin representations $K_n^{(1)}|_{Spin(n)}, K_n^{(2)}|_{Spin(n)}$ are equivalent.

We now want to show that for n even $\tilde{K}_n = K_n|_{Spin(n)}$ splits in two (irreducible) representations:

Lemma: For $n = 2m$ we get $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$, where

$$\Delta_n^{\pm} := \{ x \in \Delta_n \mid (i)^m K_n(w_\nu)(x) = \pm x \}$$ are

κ_n-invariant subspaces and therefore define themselfes representations. Elements in $\Delta_n^{\pm}$ are called Weyl-spinors.

Pf: We know that $w^2 = (-1)^m$ and $w e_j = -e_j w$

With $\phi := (i)^m K_n(w) : \Delta_n \to \Delta_n$ we get $\phi^2 = (-1)^m K_n(w^2) = id_{\Delta_n}$

and $\phi \circ \kappa_n(u) = (i)^m \underbrace{\kappa_n(wu)}_{\kappa_n(uw)} = \kappa(u) \circ \phi$ for $u \in Spin(n)$

we get the claim. ∎

We close the discussion of representations by noting:

Lemma: The representations $\kappa_n^{\pm} : Spin(n) \to \Delta_n^{\pm}$ for n even and $\kappa_n : Spin(n) \to \Delta_n$ for n odd are all irreducible

Def: (Clifford Multiplication): The map $\mu : \mathbb{R}^n \otimes_{\mathbb{R}} \Delta_n \to \Delta_n$

$X \otimes \varphi \mapsto \mu(X \otimes \varphi) := \kappa_n(X)\varphi := X \cdot \varphi$ is called Clifford multiplication

Lemma: Clifford Multipl. is ~~equiv.~~ Spin-equiv., i.e. $\forall u \in Spin(n)$, $X \in \mathbb{R}^n$, $\varphi \in \Delta_n$ we have $\kappa(u)(\mu(X \otimes \varphi)) = \mu(\varrho(u)X \otimes \kappa(u)\varphi)$
and ~~...~~ κ is unitary (on $Spin(n)$)

Spin structures on manifolds

Let $(P, \pi_P, M, SO(n))$ be the principal $SO(n)$-bundle of oriented ~~orthogonal~~ orthonormal frames for the tangent bundle of the oriented Riemannian manifold M^n.

Def: A spin structure of a Riemannian manifold M is a principal $Spin(n)$-bundle $(Q, \pi_Q, M, Spin(n))$ together with a C^∞-map $f : Q \to P$ s.t. the following diagram commutes

$$\begin{tikzcd}
Q \times Spin(n) \arrow[r] \arrow[d, "f \times \varrho"] \arrow[dr, phantom, "\text{right-actions}"] & Q \arrow[d, "f"] \arrow[dr, "\pi_Q"] & \\
P \times SO(n) \arrow[r] & P \arrow[r, "\pi_P"] & M
\end{tikzcd}$$

We say that M is a spin manifold if it admits a spinstructure

Def: The associated vector-bundle $S := Q \times_{(\kappa)} \Delta_n$ is called the spin-bundle of $(M, g, (Q, f))$. It is a complex vectorbundle of rank $2^{[\frac{n}{2}]}$ and ~~elem.~~ sections $s \in \Gamma(S)$ are called spinorfields.

Lifting a metric connection, we get a spin connection $\to$ S is a Clifford bundle

Not every manifold admits a spinstructure and the obstruction for M to be spin turns out to be a purely topological one, measured by the 2nd Stiefel-Whitney classes: For this, we make a

Quick Intermezzo on Čech cohomology:

Let $\{U_i\}_{i \in A}$ be a good cover of ~~the~~ the manifold M^n, i.e.

$U_{\alpha_1} \cap U_{\alpha_2} \dots \cap U_{\alpha_k}$ is either empty or diff. to $\mathbb{R}^n$ ($\rightsquigarrow$ will make Čech-cohom. isomorphic to de-Rham-coh.)

Def: a Čech r-cochain is a function $f(i_0, i_1, \dots, i_r) \in \mathbb{Z}_2$ defined on $U_{i_0} \cap U_{i_1} \dots U_{i_r} \neq \emptyset$ which is totally symmetric under an arbitrary permutation of its indices.

Denote $C^r(M, \mathbb{Z}_2)$ the multiplicative group of Čech-r-~~chains~~ cochains.

9

Def: The coboundary operator is defined by $\delta : C^r(M,\mathbb{Z}_2) \to C^{r+1}(M,\mathbb{Z}_2)$

$$(\delta f)(i_0,\dots i_{r+1}) = \prod_{j=0}^{r+1} f(i_0,\dots,\hat{i}_j \dots, i_{r+1})$$

e.g. $(\delta f_0)(i_0,i_1) = f_0(i_1) f_0(i_0)$ for $f \in C^0(M,\mathbb{Z}_2)$

clearly $\delta^2 f = 1$

Cocycle group: $Z^r(M,\mathbb{Z}_2) = \{ f \in C^r(M,\mathbb{Z}_2) \mid \delta f = 1 \}$

Coboundary group: $B^r(M,\mathbb{Z}_2) = \{ f \in C^r \mid f = \delta f', f' \in C^{r-1} \}$

Def: The rth Čech-cohomology group $H^r(M,\mathbb{Z}_2)$ is defined by

$$H^r = Z^r / B^r$$

Stiefel-Whitney class

Recall that principal G-bundles can be given in terms of open coverings $\{U_i\}$ of the base manifold together with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$

s.t. $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathbb{I}$ (cocycle condition)

In that sense, a ~~spin structure of a manifold M is given by~~ manifold M has a spin structure iff there is a lifting of the $SO(n)$-~~str~~ transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n)$ to some $spin(n)$-transition functions

$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(n)$ s.t. $\rho \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$

Locally this always exists and $\rho(\tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} \tilde{g}_{\gamma\alpha}) = g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathbb{1}$

$\hookrightarrow$ $\tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} \tilde{g}_{\gamma\alpha} \in \ker \rho = \{\pm 1\}$

For the $\tilde{g}$'s to define a spinstructure, we need $\tilde{g}\tilde{g}\tilde{g} = +1$

Define the Čech-2-~~Co~~cochain $f : U_i \cap U_j \cap U_k \to \mathbb{Z}_2$ by

$\tilde{g}_{ij} \tilde{g}_{jk} \tilde{g}_{ki} = f(i,j,k) \cdot \mathbb{I}$

- f symmetric:
- f closed

$\delta f(i,j,k,l) =$

$\Rightarrow$ f defines Element $w_2(M) \in H^2(M,\mathbb{Z}_2)$