SYMPLECTIC TOPOLOGY AND HOLOMORPHIC CURVES, WINTERSEMESTER 2018–2019, HU BERLIN

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This is not a set of lecture notes, but merely a brief summary of the contents of each lecture, with reading suggestions and a compendium of exercises. The suggested reading will usually not correspond precisely to what was covered in the lectures, but there will often be a heavy overlap.

1. Introduction to symplectic topology (16.10.2018)

Topics and reading. A large portion of the contents of this lecture appear in §1.1–1.3 and §1.5 of [Wena]. For a more general basic introduction to symplectic geometry, the book [CdS01] is very popular.

- Newton's laws of motion and Hamilton's equations
- the standard symplectic form ω_{st} on $\mathbb{R}^{2n}$ (see Exercise 1.1)
- symplectic forms and Hamiltonian vector fields
- Darboux's theorem (proof postponed until next week)
- Hamiltonian flows conserve energy and preserve volume—moreover, they are *symplectomorphisms* (cf. Prop. 1.2.2 and Cor. 1.2.3 in [Wena])
- Examples of symplectic manifolds: $\mathbb{R}^{2n}$, $\mathbb{T}^{2n}$, oriented surfaces, products, $\mathbb{CP}^n$ (mentioned with details deferred; see [Wen18, Example 1.4]), and why S^{2n} is not one unless $n = 1$ (de Rham cohomology)
- the canonical symplectic form on a cotangent bundle (see Exercise 1.2)
- Questions/results in symplectic *topology*:
  1. (open question) If T^*M and T^*N are symplectomorphic, must M and N be diffeomorphic?
  2. (Gromov [Gro85]) Every symplectic form on $\mathbb{R}^4$ that is standard near infinity is symplectomorphic to the standard one (cf. [Wena, Theorem 1.5.1]). Sketch of proof via J-holomorphic curves.

Exercises.

Exercise 1.1. Recall that a differential form ω on a manifold M is called **closed** if its exterior derivative $d\omega$ vanishes. If ω is a 2-form, it is called **nondegenerate** if there does not exist any point $p \in M$ with a nontrivial tangent vector $X \in T_pM$ such that $\omega(X, Y) = 0$ for all $Y \in T_pM$.

(a) Show that a 2-form ω is nondegenerate if and only if for every $p \in M$, the natural linear map $T_pM \to T_p^*M : X \mapsto \omega(X, \cdot)$ is an isomorphism.

(b) Prove that on $\mathbb{R}^{2n}$ with coordinates $(p_1, q_1, \dots, p_n, q_n)$, the 2-form

$$\omega_{\mathrm{st}} := \sum_{j=1}^{n} dp_j \wedge dq_j$$

is closed and nondegenerate.

(c) Prove that for any smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$, a path $x(t) = (q(t), p(t)) \in \mathbb{R}^{2n}$ satisfies Hamilton's equations of motion

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

if and only if $\dot{x}(t) = X_H(x(t))$, where X_H is the unique vector field on $\mathbb{R}^{2n}$ satisfying

$$\omega_{\mathrm{st}}(X_H, \cdot) = -dH.$$

(Note that this vector field exists and is unique due to the isomorphism in part (a) and the fact that ω_{st} is nondegenerate.)

Exercise 1.2. Given a smooth n-manifold M, the **canonical 1-form** $\lambda_{\mathrm{can}} \in \Omega^1(T^*M)$ on the cotangent bundle is defined by $\lambda_{\mathrm{can}}(\xi) = \tau'(\xi)\,(T\pi(\xi))$, where

$$\pi : T^*M \to M, \qquad \tau : TM \to M, \qquad \tau' : T(T^*M) \to T^*M$$

denote the natural bundle projections and $T\pi : T(T^*M) \to TM$ is the tangent map of π.

(a) Suppose $\mathcal{U} \subset M$ is an open subset admitting a coordinate chart $(q^1, \ldots, q^n)$, and the induced coordinates on $T^*M|_{\mathcal{U}} \subset T^*M$ are denoted by $(q^1, \ldots, q^n, p^1, \ldots, p^n)$. (This means concretely that if $x \in \mathcal{U}$ denotes the point with coordinate values $(q^1, \ldots, q^n)$, then the coordinate values $(q^1, \ldots, q^n, p^1, \ldots, p^n)$ represent the cotangent vector $p^1\,dq^1 + \ldots + p^n\,dq^n$ in $T^*_x M$.) Prove that

$$\lambda_{\mathrm{can}} = \sum_{j=1}^n p^j\,dq^j \quad \text{on} \quad T^*M|_{\mathcal{U}}.$$

Conclude that $d\lambda_{\mathrm{can}}$ is symplectic and that the 1-form $\sum_j p^j\,dq^j$ is independent of the original choice of coordinate chart $(q^1, \ldots, q^n)$.

(b) Suppose $\langle\ ,\ \rangle$ is a Riemannian metric on M, and use the same notation to denote the inner product on cotangent spaces $T^*_q M$ induced via the isomorphism $T_q M \to T^*_q M : X \mapsto \langle X, \cdot\rangle$. With this understood, denote elements of T^*M by (q, p) for $q \in M$ and $p \in T^*_q M$, and consider the Hamiltonian function $H : T^*M \to \mathbb{R}$ defined by

$$H(q,p) = \frac{1}{2}\langle p, p\rangle.$$

Show that a path $x(t) = (q(t), p(t))$ in T^*M satisfies $\dot{x} = X_H(x)$ if and only if $p(t) = \langle \dot{q}(t), \cdot\rangle$ and $t \mapsto q(t)$ is a geodesic on M with respect to the metric $\langle\ ,\ \rangle$.

Hint 1: It helps to think in variational terms. Convince yourself first that on any exact symplectic manifold $(W, d\lambda)$ with any function $H : W \to \mathbb{R}$, a trajectory $x : [t_0, t_1] \to W$ is an orbit of X_H if and only if it is a stationary point of the functional $\mathcal{A}(x) := \int_{t_0}^{t_1} [\lambda(\dot{x}(t)) - H(x(t))]\,dt$, defined on the space of smooth paths $x : [t_0, t_1] \to W$ with fixed end points. Then write down this functional explicitly for paths $x(t) = (q(t), p(t)) \in T^*M$ with $H(q,p) = \frac{1}{2}\langle p, p\rangle$, and derive another characterization of its stationary points.

Hint 2: Any choice of connection on the vector bundle $\pi : T^*M \to M$ provides a convenient identification of each tangent space $T_{(q,p)}(T^*M)$ with $T_q M \oplus T^*_q M$ if you think of these two factors as containing the horizontal and vertical parts respectively of tangent vectors.

Agenda for the Übung (19.10.2018). Aside from the two exercises above, we will discuss the standard (Fubini-Study) symplectic form on $\mathbb{CP}^n$ and the consequence that every smooth projective variety is naturally a symplectic manifold.

2. Basics on symplectic manifolds (23.10.2018)

Topics and reading. The Moser deformation trick is covered in [Wena, §1.4], and you'll find a more comprehensive discussion (including complete proofs of the Moser stability and Lagrangian neighborhood theorems) in [MS17, Chapter 3]. For almost complex structures and compatibility/tameness, I recommend skimming [Wena, §2.2].

- Why the symplectomorphism group $\mathrm{Symp}(M, \omega)$ is infinite-dimensional
- Darboux's theorem and proof via the Moser deformation trick
- Moser's stability theorem (similar proof)
- The Fubini-Study symplectic form and symplectic deformations on $\mathbb{CP}^n$
- Lagrangian neighborhood theorem (stated without proof)
- Almost complex structures, tameness and compatibility
- Proof that the space $\mathcal{J}(M, \omega)$ of compatible almost complex structures is nonempty and contractible
- Corollary: For any $J_0, J_0 \in \mathcal{J}(M, \omega)$, the vector bundles (TM, J_0) and (TM, J_1) are isomorphic.

Exercises.

Exercise 2.1. Work through the details of [Wen18, Example 1.4] until you understand the definition of the Fubini-Study symplectic form ω_{FS} on $\mathbb{CP}^n$. Then prove:

(a) Every complex submanifold $\Sigma \subset \mathbb{CP}^n$ is also a symplectic submanifold of $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$, i.e. the restriction of ω_{FS} to Σ is also symplectic.
Hint: Since $\mathbb{CP}^n$ is a complex manifold, it has a natural almost complex structure defined as multiplication by i in any local holomorphic coordinates. Show that this almost complex structure is tamed by ω_{FS}. (Why does that help?)

(b) For each $k \leqslant n$, the embedding $\iota : \mathbb{CP}^k \hookrightarrow \mathbb{CP}^n : [z_0 : \ldots : z_k] \mapsto [z_0 : \ldots : z_k : 0 : \ldots : 0]$ satisfies $\iota^* \omega_{\mathrm{FS}} = \omega_{\mathrm{FS}}$.

(c) $\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}} = \pi$. *Hint: Find an embedding $\varphi : \mathbb{C} \hookrightarrow S^3$ such that for the projection $\mathrm{pr} : S^3 \to \mathbb{CP}^1 = S^3/S^1$, $\mathrm{pr} \circ \varphi$ is a diffeomorphism of $\mathbb{C}$ to the complement of a point in $\mathbb{CP}^1$. Then use the relation $\mathrm{pr}^* \omega_{\mathrm{FS}} = \omega_{\mathrm{st}}|_{TS^3}$ to integrate $(\mathrm{pr} \circ \varphi)^* \omega_{\mathrm{FS}}$ over $\mathbb{C}$.*

Exercise 2.2. Let $x_0 = [1 : 0 : 0] \in \mathbb{CP}^2$, and consider the holomorphic map

$$\pi : \mathbb{CP}^2 \backslash \{x_0\} \to \mathbb{CP}^1 : [z_0 : z_1 : z_2] \mapsto [z_1 : z_2].$$

Show that the closure of each level set $\pi^{-1}(\mathrm{const}) \subset \mathbb{CP}^2$ can be parametrized by a holomorphic embedding $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ that passes through x_0, thus it defines a complex submanifold $\Sigma \subset \mathbb{CP}^2$ which is diffeomorphic to S^2.

3. J-holomorphic curves and the linearized Cauchy-Riemann operator (30.10.2018)

Topics and reading. For a readable introduction to the first Chern class in the symplectic context, see [MS17, §2.7]. (If you want to delve more deeply into the subject of characteristic classes, try [Hat] or [MS74], though keep in mind that one can also make an entire course out of that subject on its own.) Other than this, most of the contents of this week's lecture (in particular the derivation of the linearized Cauchy-Riemann operator) are covered in [Wena, §2.3–2.4].

- Axiomatic description of the first Chern class on complex vector bundles and $c_1(M, \omega) := c_1(TM, J)$ for $J \in \mathcal{J}(M, \omega)$
- $\langle c_1(T\Sigma), [\Sigma] \rangle = \chi(\Sigma)$ for closed oriented surfaces (Poincaré-Hopf theorem)
- Sketch of proof that $c_1(\mathbb{CP}^2, \omega_{\mathrm{FS}}) = 3e$ for the standard generator $e \in H^2(\mathbb{CP}^2)$

- Almost complex manifolds, pseudoholomorphic curves and the nonlinear Cauchy-Riemann equation
- Riemann surfaces = almost complex 2-manifolds = complex 1-manifolds (lemma of Gauss)
- The nonlinear Cauchy-Riemann operator $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$ as a section of an infinite-dimensional vector bundle $\mathcal{E} \to \mathcal{B}$
- Linearizaion at zeroes and the implicit function theorem (implying $\bar{\partial}_J^{-1}(0)$ is a smooth manifold near u if $D\bar{\partial}_J(u) : T_u\mathcal{B} \to \mathcal{E}_u$ is surjective with a continuous right inverse)
- Computation of $\mathbf{D}_u := D\bar{\partial}_J(u) : \Gamma(u^*TM) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$, i.e. for any symmetric connection ∇ on M,

$$\mathbf{D}_u\eta = \nabla\eta + J(u) \circ \nabla\eta \circ j + (\nabla_\eta J) \circ Tu \circ j \tag{3.1}$$

- Statement of the index theorem for $\mathbf{D}_u$

Exercises.

Exercise 3.1. For any choice of area forms ω_1, ω_2 and complex structures[1] j_1, j_2 on S^2 such that all are compatible with the standard orientation of S^2, it is easy to show that the product almost complex structure $J := j_1 \oplus j_2$ on $S^2 \times S^2$ is compatible with the product symplectic form $\omega := \omega_1 \oplus \omega_2$. By the Künneth formula, $H_2(S^2 \times S^2) \cong \mathbb{Z}^2$ is generated by the two elements $e_1, e_2 \in H_2(S^2 \times S^2)$ represented by oriented submanifolds of the form $S^2 \times \{\text{const}\}$ and $\{\text{const}\} \times S^2$. Following the same procedure we used in lecture for computing $c_1(\mathbb{CP}^2, \omega_{\mathrm{FS}})$, show that

$$\langle c_1(S^2 \times S^2, \omega), e_1 \rangle = \langle c_1(S^2 \times S^2, \omega), e_2 \rangle = 2.$$

This uniquely determines $c_1(S^2 \times S^2, \omega)$ since $H^2(S^2 \times S^2) \cong \mathrm{Hom}(H_2(S^2 \times S^2), \mathbb{Z})$ by the universal coefficient theorem.

Exercise 3.2. Show that the operator $\mathbf{D}_u : \Gamma(u^*TM) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ defined in (3.1) for any J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ satisfies the following variation on the usual Leibniz rule for covariant derivatives:

$$\mathbf{D}_u(f\eta) = (\bar{\partial} f)\eta + f\mathbf{D}_u\eta \quad \text{ for all } \eta \in \Gamma(u^*TM) \text{ and } f \in C^\infty(\Sigma, \mathbb{R}),$$

where we associate to every smooth function $f : \Sigma \to \mathbb{C}$ the complex-valued 1-form $\bar{\partial} f := df + i\, df \circ j$ which vanishes if and only if f is a holomorphic function. (Note however that the values of f in our Leibniz rule are required to be real, not complex.)

Agenda for the Übung (2.11.2018). This week's Übung will not discuss exercises but will instead be an extra lecture to cover background material from functional analysis, including as much as possible of the following:

- Compact and Fredholm operators on Banach spaces (my favorite references for this material are [Tay96, Appendix A] and [AA02], but there are plenty of other good options)
- Differential calculus in Banach spaces, Banach manifolds and the inverse/implicit function theorem in infinite dimensions (good references for this material are [Lan93, Chapters XIII–XIV] and [Lan99, Chapters I–III])
- A crash course in distributions and Sobolev spaces (see [Wena, §2.5] and [Wenb, Appendix A])

[1] I am omitting the word "almost" here since S^2 is a surface, so Gauss tells us that an almost complex structure is equivalent to a complex manifold structure.

4. Elliptic regularity, part 1 (6.11.2018)

Topics and reading. A quick overview of most of this week's topics may be found in [Wenb, §2.3], but you might also want to look at §2.5 of [Wena] for background on convolutions and Fourier transforms of distributions, and §2.6 for a more detailed treatment of the fundamental elliptic estimate and the bounded right-inverse for $\bar{\partial}$. If you enjoy that stuff too much and have unlimited free time, you'll find the "hard" part of the proof of the main estimate (i.e. the Calderón-Zygmund inequality) in Appendix 2.A of [Wena] and a more general discussion of elliptic operators in Appendix 2.B. Finally, you will find some extra lecture notes on Sobolev spaces at

http://www.mathematik.hu-berlin.de/~wendl/Sobolev.pdf

which include (in §A.2) a detailed explanation of the norm on $W^{k,p}(E)$ and its independence of the various choices. (These notes were written to be a new appendix for a revision of [Wena] that is not yet finished, but a condensed version of this appendix also appears in [Wenb].)

- Linear Cauchy-Riemann type operators on vector bundles
- The Sobolev space of sections $W^{k,p}(E)$ of a vector bundle $\pi : E \to \Sigma$ (see §A.2 in the notes on Sobolev spaces mentioned above)
- Precise statement of the Fredholm theorem for Cauchy-Riemann type operators $\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E))$ with $k \in \mathbb{N}$ and $1 < p < \infty$
- The bounded right-inverse of $\bar{\partial} : W^{1,p}(\mathbb{D}) \to L^p(\mathbb{D})$ and the fundamental elliptic estimate for $\bar{\partial}$
- Construction of the right-inverse via fundamental solution; Fourier transform argument for the case $p = 2$

Exercises.

Exercise 4.1. This exercise concerns linear Cauchy-Riemann type operators on complex vector bundles E over Riemann surfaces (Σ, j).

(a) Show that if $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ is any linear Cauchy-Riemann type operator, then every other linear Cauchy-Riemann type operaor on E is of the form $\mathbf{D}' = \mathbf{D} + A$ for some smooth real-linear bundle map (i.e. a "zeroth-order" term) $A : E \to F$.
Hint: Show that if $A := \mathbf{D}' - \mathbf{D} : \Gamma(E) \to \Gamma(F)$ then A is C^∞-linear, i.e. $A(f\eta) = fA\eta$ for all $f \in C^\infty(\Sigma, \mathbb{R})$ and $\eta \in \Gamma(E)$.

(b) Show that by choosing suitable local coordinates and local trivializations, every linear Cauchy-Riemann type operator can be identified in a neighborhood of any given point with an operator of the form $\bar{\partial} + A : C^\infty(\mathbb{D}, \mathbb{C}^m) \to C^\infty(\mathbb{D}, \mathbb{C}^m)$, where $\bar{\partial} := \partial_s + i\partial_t$ in the standard coordinates $s + it$ on the unit disk $\mathbb{D} \subset \mathbb{C}$, and $A \in C^\infty(\mathbb{D}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m))$.

Exercise 4.2. Define a function $K : \mathbb{C} \to \mathbb{C}$ almost everywhere by $K(z) = 1/2\pi z$.

(a) Prove that K is in $L^1_{\mathrm{loc}}(\mathbb{C})$ and $\bar{\partial} K = \delta$ in the sense of distributions, where δ is the distribution defined by $\langle \delta, \varphi \rangle = \varphi(0)$ for test functions φ, and $\bar{\partial} := \partial_s + i\partial_t$ in coordinates $s + it$ on $\mathbb{C}$.

(b) Prove that for $\partial := \partial_s - i\partial_t$, ∂K is the distribution defined on test functions φ by

$$\langle \partial K, \varphi \rangle = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{\mathbb{C} \setminus \mathbb{D}_\epsilon} \frac{\varphi(z)}{z^2} \, d\mu(z),$$

where $\mathbb{D}_\epsilon \subset \mathbb{C}$ denotes the disk of radius ϵ and $d\mu(z)$ is the Lebesgue measure for integrating functions of $z \in \mathbb{C}$. (Note that informally, this just says $\partial K = 2\frac{d}{dz}K = -1/\pi z^2$, but the latter is not a locally integrable function on $\mathbb{C}$, so the limiting process is necessary in order to define it as a distribution.)

Note: You'll find these exercises worked out in detail in [Wena, Prop. 2.6.12 and Lemma 2.6.14], *but you might enjoy trying them youself first.*

5. Elliptic regularity, part 2 (13.11.2018)

Topics and reading. Everything for the Tuesday lecture this week is contained in [Wena, §2.6], and a more concise treatment of the same material can also be found in [Wenb, §2.4.1] (especially Theorem 2.16). For more details on difference quotients and the use of the Banach-Alaoglu theorem on weak convergence, see §A.1.4 in the extra notes on Sobolev spaces at http://www.mathematik.hu-berlin.de/~wen
The Übung this week was on Wednesday and was essentially a continuation of the Tuesday lecture: its contents are covered (in much more detail) in §3.2–3.3 (for formal adjoints and the Fredholm property) and §2.7–2.8 (for local existence and the similarity principle) of [Wena].

- Local regularity theorem for solutions of $\bar{\partial} u = f$ (see [Wenb, Theorem 2.16])
- Difference quotients and the Banach-Alaoglu theorem
- Corollary: If $u \in W^{1,p}(\mathbb{D}, \mathbb{C}^m)$ and $(\bar{\partial} + A)u = 0$ for some $A \in C^\infty(\mathbb{D}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m))$ then u is smooth
- If $u \in L^1$ and $\bar{\partial} u = 0$ weakly then u is smooth; proof via mean value property for harmonic functions
- Corollary: If $u \in L^p(\mathbb{D})$ and $\bar{\partial} u \in W^{k,p}(\mathbb{D})$ then u is of class $W^{k+1,p}$ on any smaller disk

Exercises.

Exercise 5.1. As a corollary of what we proved in lecture about weak regularity for the equation $\bar{\partial} u = f$, prove that for any $p \in (1, \infty)$ and $A \in C^\infty(\mathbb{D}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m))$, if $u \in L^p(\mathbb{D}, \mathbb{C}^m)$ is a weak solution to the equation $(\bar{\partial} + A)u = f$ for some $f \in W^{k,p}(\mathbb{D}, \mathbb{C}^m)$, then u is of class $W^{k+1,p}$ on $\mathbb{D}_r$ for any $r < 1$.

5.1. **Übung (14.11.2018).** The Übung this week was a continuation of the lecture, and covered the following topics:

- The global estimate $\|\eta\|_{W^{k,p}} \leqslant c\|\mathbf{D}\eta\|_{W^{k,p}} + c\|\eta\|_{W^{k-1,p}}$ for Cauchy-Riemann type operators $\mathbf{D}$ on a complex vector bundle $E \to \Sigma$ over a closed Riemann surface
- Functional-analytic criterion for an operator $\mathbf{D} \in \mathscr{L}(X, Y)$ to have finite-dimensional kernel and closed image (see [Wena, Prop. 3.3.3])
- The formal adjoint of a Cauchy-Riemann type operator and the splittings $W^{k-1,p}(F) = \operatorname{im} \mathbf{D} \oplus \ker \mathbf{D}^*$, $W^{k-1,p}(E) = \operatorname{im} \mathbf{D}^* \oplus \ker \mathbf{D}$ (proof via the Hahn-Banach theorem using weak regularity)
- Proof that Cauchy-Riemann type operators and their formal adjoints are Fredholm
- Local existence result for solutions to $(\bar{\partial} + A)u = 0$ with $A \in L^p(\mathbb{D}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m))$ and $2 < p < \infty$ (see [Wena, §2.7]
- Corollary 1: Complex-linear Cauchy-Riemann type operators are equivalent to holomorphic vector bundle structures
- Corollary 2 (the similarity principle): solutions to $(\bar{\partial} + A)\eta = 0$ look locally like holomorphic functions in some continuous trivialization
- Every Cauchy-Riemann type operator $\mathbf{D}$ on a complex line bundle $E \to \Sigma$ over a closed Riemann surface satisfying $c_1(E) := \langle c_1(E), [\Sigma] \rangle < 0$ is injective.

6. Riemann-Roch and nonlinear regularity (21.11.2018)

Topics and reading. The proof of the Riemann-Roch formula for the genus zero case in this lecture is explained in [Wena, §3.4]. For the nonlinear regularity theorem (stated mostly without proof in the lecture), a mostly complete proof may be found in [Wenb, §2.4.2]; a slightly different version with more details is also in [Wena, §2.11].

- The Riemann-Roch formula $\operatorname{ind}(\mathbf{D}) = (\operatorname{rank}_{\mathbb{C}} E)\chi(\Sigma) + 2c_1(E)$, and why it suffices to prove it for $\operatorname{rank}_{\mathbb{C}} E = 1$
- Every Cauchy-Riemann type operator $\mathbf{D}$ on a complex line bundle $E \to \Sigma$ over a closed Riemann surface satisfying $c_1(E) > -\chi(\Sigma)$ is surjective.
- Corollary: On line bundles over S^2, every Cauchy-Riemann type operator is either injective or surjective (and both if $c_1(E) = -1$).
- Proof of Riemann-Roch for line bundles over S^2 (see Exercise 6.2 below)
- Quick sketch of Taubes's proof of Riemann-Roch via large antilinear deformations (see [Wenb, Lecture 5])
- Nonlinear regularity theorem for local J-holomorphic curves (see [Wenb, Theorem 2.22])

Exercises.

Exercise 6.1. Show that every complex vector bundle E over a surface Σ is isomorphic to a direct sum of complex line bundles.
Hint: If $\operatorname{rank}_{\mathbb{C}} E > 1$*, then by standard transversality results, every smooth section of* E *can be perturbed to one that is nowhere zero. (Why?)*

Exercise 6.2. For any integer $k \geqslant 0$, let E_k^α and E_k^β denote two copies of the trivial complex line bundle $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, and define

$$E_k := \left(E_k^\alpha \amalg E_k^\beta \right) / \sim,$$

where the equivalence relation identifies $(z, v) \in E_k^\alpha$ with $(1/z, (1/z^k)v) \in E_k^\beta$ for each $z \in \mathbb{C} \backslash \{0\}$. Identifying S^2 with the extended complex plane $\mathbb{C} \cup \{\infty\}$, define a projection

$$\pi : E_k \to S^2$$

by $\pi(z, v) = z$ for $(z, v) \in E_k^\alpha$ and $\pi(z, v) = 1/z$ for $(z, v) \in E_k^\beta$.

(a) Construct a holomorphic vector bundle structure for $\pi : E_k \to S^2$ such that all holomorphic sections $\eta : S^2 \to E_k$ restrict to $\mathbb{C}$ and $S^2 \backslash \{0\}$ as holomorphic sections of the trivial bundles E_k^α and E_k^β respectively.
(b) Show that a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ (viewed as a holomorphic section of E_k^α) extends over S^2 as a holomorphic section of E_k if and only if the function $g(z) := z^k f(1/z)$ extends holomorphically to $z = 0$, and that the set of functions satisfying this condition is the set of all complex polynomials of degree at most k.
(c) Show that for any of the nontrivial holomorphic sections $\eta \in \Gamma(E_k)$ in part (b), the algebraic count of the zeroes of η is k.

Part (c) proves $c_1(E_k) = k$, so the lemma we proved in lecture about Cauchy-Riemann type operators on line bundles implies that the standard Cauchy-Riemann operator $\bar\partial$ on this bundle is surjective. By part (b), $\ker \bar\partial$ is a complex vector space of dimension $1 + k$, so its real dimension is $2 + 2k = \chi(S^2) + 2c_1(E_k)$, exactly what is predicted by the Riemann-Roch formula.

Scheduling note: There is no Übung either this week or next week; the next scheduled Übung is on December 7 in the usual time and place.

7. Moduli spaces and Teichmüller space (27.11.2018)

Topics and reading. The topics of this week's lecture are covered mainly in [Wena, §4.1–4.2], up to the top of page 156 (which is where we will begin next week). Exceptions: We did not yet state the main theorems about smoothness of $\mathcal{M}(J)$ and transversality for generic J, which take up the second half of §4.1, and [Wena] does not say anything about pair-of-pants decompositions, but you can find more on that topic in [Wenb, §9.3.3]. (We will come back to it later when we discuss compactness.)

- Definition of the moduli space $\mathcal{M}(J) := \mathcal{M}_{g,m}(A,J)$ of unparametrized J-holomorphic curves of genus $g \geqslant 0$ with $m \geqslant 0$ marked points homologous to $A \in H_2(M)$ in an almost complex manifold (M,J)
- The evaluation map $\mathrm{ev} : \mathcal{M}_{g,m}(A,J) \to M^{\times m} : [(\Sigma,j,\Theta,u)] \mapsto (u(\zeta_1),\dots,u(\zeta_m))$ for $\Theta = (\zeta_1,\dots,\zeta_m)$
- Definition of convergence in $\mathcal{M}(J)$
- The moduli space of Riemann surfaces $\mathcal{M}_{g,m}$ (i.e. the case $M = \{\mathrm{pt}\}$)
- The automorphism group $\mathrm{Aut}(\Sigma,j,\Theta)$ for $[(\Sigma,j,\Theta)] \in \mathcal{M}_{g,m}$
- Statement of the uniformization theorem (without proof)
- Concrete descriptions of $\mathcal{M}_{0,m}$ for $m \geqslant 0$ and $\mathcal{M}_{1,0}$, with the corresponding automorphism groups
- Definition of **stable** Riemann surfaces (with marked points)
- Theorem (not yet proved): In stable cases ($2g + m \geqslant 3$), $\mathrm{Aut}(\Sigma,j,\Theta)$ is finite and $\mathcal{M}_{g,m}$ is a smooth orbifold of dimension $6g - 6 + 2m$ with local isotropy groups $\mathrm{Aut}(\Sigma,j,\Theta)$
- Singular pair-or-pants decompositions and Fenchel-Nielsen coordinates (a sketch)
- $\mathcal{M}_{g,m} \cong \mathcal{J}(\Sigma)/\operatorname{Diff}_+(\Sigma,\Theta)$
- The subgroup $\operatorname{Diff}_0(\Sigma,\Theta)$ acts *freely* on $\mathcal{J}(\Sigma)$; proof via the Lefschetz fixed point theorem
- Definition of the **Teichmüller space** $\mathcal{T}(\Sigma,\Theta)$; $\mathcal{M}_{g,m}$ as the quotient of $\mathcal{T}(\Sigma,\Theta)$ by the (discrete) mapping class group $\operatorname{Diff}_+(\Sigma,\Theta)/\operatorname{Diff}_0(\Sigma,\Theta)$
- Theorem (not yet proved): $\mathcal{T}(\Sigma,\Theta)$ is a smooth manifold, and for any $[(\Sigma,j,\Theta)] \in \mathcal{M}_{g,m}$, $\dim \mathcal{T}(\Sigma,\Theta) - \dim \mathrm{Aut}(\Sigma,j,\Theta) = 6g - 6 + 2m$.

I failed to give any exercises this week but will make up for it next time.

8. Fredholm regular curves in the moduli space (4.12.2018)

Topics and reading. A more detailed presentation of the contents of this week's lecture can be found in [Wena, §4.2–4.3].

- Banach manifold setup for analysis of $\mathrm{Aut}(\Sigma,j,\Theta)$ and Teichmüller space $\mathcal{T}(\Sigma,\Theta) = \mathcal{J}(\Sigma)/\operatorname{Diff}_0(\Sigma,\Theta)$
- The Cauchy-Riemann type operator $\mathbf{D}_{(j,\Theta)} : W^{k,p}_\Theta(T\Sigma) \to W^{k-1,p}(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ and natural isomorphisms $\ker \mathbf{D}_{(j,\Theta)} = T_{\mathrm{Id}} \mathrm{Aut}(\Sigma,j,\Theta)$ and $\operatorname{coker} \mathbf{D}_{(j,\Theta)} = T_{[j]}\mathcal{T}(\Sigma,\Theta)$
- Teichmüller slices: definition, existence, and invariance under holomorphic group actions
- Banach manifold setup for analyzing a neighborhood of $[(\Sigma,j_0,\Theta,u_0)]$ in $\mathcal{M}_{g,m}(A,J)$
- Fredholm regular curves
- Theorem: the open set of Fredholm regular curves in $\mathcal{M}_{g,m}(A,J)$ is a smooth orbifold (with local isotropy groups $\mathrm{Aut}(u)$) whose dimension equals its **virtual dimension**
$$\operatorname{vir-dim} \mathcal{M}_{g,m}(A,J) := (n-3)(2-2g) + 2c_1(A) + 2m,$$
where $2n$ is the dimension of the target almost complex manifold (M,J), and $c_1(A) := \langle c_1(TM,J), A\rangle \in \mathbb{Z}$.

Exercises.

Exercise 8.1. Suppose (Σ,j) is a Riemann surface with a finite subset $\Theta \subset \Sigma$, and ∇ denotes the Levi-Civita connection on $T\Sigma$ with respect to any Riemannian metric compatible with the conformal structure defined by j (i.e. j and the metric define the same notion of "right angles").

Show that for any smooth family of diffeomorphisms $\varphi_\tau \in \mathrm{Diff}(\Sigma, \Theta)$ parametrized by $\tau \in (-\epsilon, \epsilon)$ with $\varphi_0 = \mathrm{Id}$, if $\partial_\tau \varphi_\tau|_{\tau=0} = X \in \Gamma(T\Sigma)$, then

$$\partial_\tau \left(\varphi_\tau^* j\right)\big|_{\tau=0} = -\nabla X \circ j + j \circ \nabla X = j(\nabla X + j \circ \nabla X \circ j) = j(\bar{\partial} X),$$

where $\bar{\partial} : \Gamma(T\Sigma) \to \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ denotes the canonical Cauchy-Riemann type operator defined via the holomorphic structure of the bundle $T\Sigma \to \Sigma$.

Exercise 8.2. For an even-dimensional real vector space V, let $\mathcal{J}(V) = \{J \in \mathrm{Aut}(V) \mid J^2 = -\mathbb{1}\}$, i.e. $\mathcal{J}(V)$ is the space of all linear complex structures on V.

(a) Show that $\mathcal{J}(V)$ is a smooth submanifold of $\mathrm{Aut}(V) \cong \mathrm{GL}(2n, \mathbb{R})$, and any choice of element $J_0 \in \mathcal{J}(V)$ gives rise to a natural bijection of $\mathcal{J}(V)$ with the homogeneous space $\mathrm{Aut}(V)/\,\mathrm{Aut}_{\mathbb{C}}(V, J_0)$, given by

$$\mathrm{Aut}(V)/\,\mathrm{Aut}_{\mathbb{C}}(V, J_0) \to \mathcal{J}(V) : [A] \mapsto A j_0 A^{-1},$$

where $\mathrm{Aut}_{\mathbb{C}}(V, J_0) := \{A \in \mathrm{Aut}(V) \mid AJ_0 = J_0 A\}$.

(b) Show that for any $J \in \mathcal{J}(V)$,

$$T_J \mathcal{J}(V) = \overline{\mathrm{End}}_{\mathbb{C}}(V, J) := \{A \in \mathrm{End}(V) \mid AJ = -JA\}.$$

(c) Given $J \in \mathcal{J}(V)$ and a neighborhood $\mathcal{O} \subset \overline{\mathrm{End}}_{\mathbb{C}}(V, J)$ of 0, consider the smooth map

$$\mathcal{O} \to \mathcal{J}(V) : Y \mapsto \left(\mathbb{1} + \frac{1}{2} JY\right) J \left(\mathbb{1} + \frac{1}{2} JY\right)^{-1},$$

which is well defined if $\mathcal{O}$ is sufficiently small since $\mathbb{1} + \frac{1}{2}JY$ is invertible if Y is small enough. Show that the derivative of this map at $0 \in \mathcal{O}$ is the identity map on $\overline{\mathrm{End}}_{\mathbb{C}}(V, J) = T_J\mathcal{J}(V)$, thus by the inverse function theorem, the map identifies a neighborhood of 0 in $\mathcal{O}$ diffeomorphically with a neighborhood of J in $\mathcal{J}(V)$.

Agenda for the Übung (7.12.2018). We will discuss the two exercises above, but if there is demand for it, we can also talk about other exercises from the last few weeks. I would also like to say some things about the construction of smooth Banach manifold structures on spaces like $W^{k,p}(\Sigma, M)$ for $kp > 2$; the canonical reference for this is [Elĭ67].

9. Transversality for somewhere injective curves (11.12.2018)

Topics and reading. A complete proof of the theorem on generic transversality for somewhere injective curves may be found in [Wena, §4.4] (excluding §4.4.2, which I plan to discuss briefly next week). For the necessary background on simple curves vs. multiple covers, see [Wena, §2.15], and for the Floer C^ε-space, [Wenb, Appendix B].

- u Fredholm regular implies $\mathrm{ind}(u) \geqslant 0$, where $\mathrm{ind}(u) := \text{vir-dim}\,\mathcal{M}_{g,m}(A, J)$ for $u \in \mathcal{M}_{g,m}(A, J)$
- Example: double covers of a J-holomorphic sphere v in an 8-manifold with $c_1([v]) = -1$ must sometimes exist but can never be regular.
- Injective points, somewhere injectivity, simple curves vs. multiple covers (see [Wena, §2.15])
- Main transversality theorem: On any closed symplectic manifold (M, ω), there exists a comeager[2] subset $\mathcal{J}^{\mathrm{reg}}(M, \omega) \subset \mathcal{J}(M, \omega)$ such that for all $J \in \mathcal{J}^{\mathrm{reg}}(M, \omega)$, all somewhere injective J-holomorphic curves are regular.
- The Sard-Smale theorem for smooth Fredholm maps

[2] **comeager** := "a countable intersection of open and dense sets"; by the Baire category theorem, these are always dense if the ambient space is metrizable and complete. In the symplectic topology literature, it is also common to see the terms "Baire subset" and "set of second category" used as synonyms for "comeager subset," though technically "second category" means something slightly different.

- The Floer C^ε-space $\mathcal{J}^\varepsilon$ of perturbed almost complex structures near a reference structure $J^{\mathrm{ref}} \in \mathcal{J}(M,\omega)$
- The universal moduli space $\mathcal{M} := \{(u,J) \mid J \in \mathcal{J}^\varepsilon,\ u \in \mathcal{M}(J) \text{ somewhere injective}\}$
- Proof (via the Hahn-Banach theorem and similarity principle) that $\mathcal{M}$ is always a smooth Banach manifold
- Conclusion via Sard-Smale: the space of $J \in \mathcal{J}(M,\omega)$ for which all somewhere injective J-holomorphic curves are regular is dense

Exercises.

Exercise 9.1. Consider the following relaxation of the hypotheses for the transversality theorem we proved in lecture. Suppose (M,ω) is a symplectic manifold (not necessarily compact), $J^{\mathrm{fix}} \in \mathcal{J}(M,\omega)$, $\mathcal{U} \subset M$ is an open subset with compact closure, and let

$$\mathcal{J}(M,\omega\,;\mathcal{U},J^{\mathrm{fix}}) := \left\{ J \in \mathcal{J}(M,\omega) \;\middle|\; J \equiv J^{\mathrm{fix}} \text{ on } M\backslash\mathcal{U} \right\},$$

regarded as a topological space with the topology of C^∞-convergence. What can you prove about Fredholm regularity of J-holomorphic curves for generic choices of $J \in \mathcal{J}(M,\omega\,;\mathcal{U},J^{\mathrm{fix}})$?

Exercise 9.2. Show that if $u = v \circ \varphi : (\Sigma,j) \to (M,J)$ is the composition of a closed somewhere injective J-holomorphic curve $v : (\Sigma',j') \to (M,J)$ with a holomorphic map $\varphi : (\Sigma,j) \to (\Sigma',j')$ of degree $d \geqslant 1$ between closed Riemann surfaces, then the group $\mathrm{Aut}(u)$ of biholomorphic diffeomorphisms $\psi : (\Sigma,j) \to (\Sigma,j)$ that satisfy $u = u \circ \psi$ has order at most d. In particular, if u is somewhere injective, then its automorphism group is trivial.

Exercise 9.3. Suppose M is a closed manifold, $E \to M$ is a smooth vector bundle and $\{\varepsilon_k\}_{k=0}^\infty$ is a sequence of positive numbers with $\varepsilon_k \to 0$. The Floer C^ε-norm for smooth sections $\eta \in \Gamma(E)$ is then defined by

$$\|\eta\|_{C^\varepsilon} := \sum_{k=0}^\infty \varepsilon_k \|\eta\|_{C_k},$$

where the C_k-norm can be defined via either a choice of connection or a finite collection of local trivializations covering M (one can show that all such choices give equivalent norms since M is compact). Prove that $C^\varepsilon(E) := \{\eta \in \Gamma(E) \mid \|\eta\|_{C^\varepsilon} < \infty\}$ is then a separable Banach space with respect to the C^ε-norm.
Hint: If you get frustrated and just want to read an answer, see [Wenb, Appendix B].

Exercise 9.4. The following functional-analytic lemma proves that the linearized operator

$$\mathbf{L} : W^{1,p}(u_0^*TM) \oplus T_{J_0}\mathcal{J}^\varepsilon \to L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma,u_0^*TM) : (\eta,Y) \mapsto \mathbf{D}_{u_0}\eta + Y \circ Tu_0 \circ j_0,$$

which we proved in lecture is surjective, must have closed image. (This is a necessary step before applying the Hahn-Banach theorem as we did in lecture.) Prove that if X, Y and Z are Banach spaces, $\mathbf{T} : X \to Y$ is a Fredholm operator and $\mathbf{A} : Z \to Y$ is a bounded linear map, then the linear map

$$\mathbf{L} : X \oplus Z \to Y : (x,z) \mapsto \mathbf{T}x + \mathbf{A}z$$

has closed image.
Hint: Remember that since $\mathbf{T}$ *is Fredholm, you can write* $X = V \oplus \ker \mathbf{T}$ *and* $Y = W \oplus C$ *such that* $C \cong \operatorname{coker} \mathbf{T}$ *and* $V \xrightarrow{\mathbf{T}} W$ *is an isomorphism.*

Exercise 9.5. Under the same hypotheses as in Exercise 9.4, prove that if $\mathbf{L}$ is surjective, then the projection

$$\Pi : \ker \mathbf{L} \to Z : (x,z) \mapsto z$$

has kernel and cokernel isomorphic to the kernel and cokernel respectively of $\mathbf{T} : X \to Y$.
Comment: Since $\mathbf{D}_{u_0}$ *is Fredholm, this is what proves that the projection* $\pi : \mathcal{M} \to \mathcal{J}^\varepsilon : (u, J) \mapsto J$ *is a Fredholm map (i.e. its derivative at every point is Fredholm), so that the Sard-Smale theorem applies. It also follows that for every* $J \in \mathcal{J}^\varepsilon$ *that is a regular value of this projection, all curves* $u \in \pi^{-1}(J)$ *are Fredholm regular.*
Hint: See [Wena, Lemma 4.4.13].

Scheduling note: The Übung next week will take place on Wednesday from 15:00 to 16:30 in 1.315 (RUD25) instead of the usual Friday time. We will discuss some subset of the exercises above (especially Exercise 9.1), and presumably also something about compactness, which is the topic for next Tuesday's lecture.

10. Compactness (18.12.2018)

Topics and reading. It's a bit tricky to find a full and readable presentation of Gromov's compactness theorem in the literature. I recommend starting with [Wen18, §2.1.6], which at least contains clean statements of the main definitions and results you need to know. For a more detailed discussion of the degeneration phenomena we discussed in this lecture, see sections 9.1 and 9.3 of [Wenb]; the former introduces the Hofer lemma and the standard bubbling/rescaling trick in the context of proving the C^0-extension part of Gromov's removable singularity theorem. In reading §9.3, you need to keep in mind that the context in [Wenb] is somewhat more general than we have been discussing, so e.g. you should probably completely skip §9.3.2 (the "breaking" phenomenon is not relevant for closed J-holomorphic curves, though it *is* relevant in Floer homology, for those of you who are learning about that at the same time).

- Review of $\mathcal{M}_{0,4} \cong S^2 \backslash \{0, 1, \infty\}$ and its obvious "compactification" to S^2
- Energy $E(u) := \int_\Sigma u^*\omega$ for a J-holomorphic curve in a symplectic manifold (M, ω) with tame J
- Energy is nonnegative, zero only for constant curves, and bounded by homology for closed curves
- Statement of Gromov's removable singularity theorem (see [Wen18, Theorem 2.36] or [Wenb, §9.1]
- Uniform C^1-bounds imply C^∞_{loc}-convergence (elliptic regularity)
- The rescaling trick for a sequence $z_k \in \Sigma$ with $|du_k(z_k)| \to \infty$
- Hofer's lemma on complete metric spaces (see [Wenb, Lemma 9.4])
- Bubbling of holomorphic spheres
- Degenerating complex structures via pair-of-pants decompositions
- Understanding the three elements of $\overline{\mathcal{M}}_{0,4} \backslash \mathcal{M}_{0,4}$ in terms of degenerate pair-of-pants decompositions
- Definition of the compactified moduli space $\overline{\mathcal{M}}_{g,m}(A, J)$ of "stable nodal J-holomorphic curves of arithmetic genus g"
- Statement of Gromov's compactness theorem

Exercises.

Exercise 10.1. Show that if J is a tame almost complex structure on a symplectic manifold (M, ω) and $u : (\Sigma, j) \to (M, J)$ is a J-holomorphic curve, then the 2-form $u^*\omega$ on Σ is nonnegative (with respect to the orientation of Σ determined by its complex structure), and vanishes only at points where the first derivative of u vanishes. In particular, $E(u) := \int_\Sigma u^*\omega \geqslant 0$ always, with equality if and only if u is locally constant.

Exercise 10.2. Set $(M, J) = (S^2, i)$ and recall that any J-holomorphic sphere with three marked points has a unique parametrization that fixes $(0, 1, \infty)$ as the ordered set of marked points. It follows that the moduli space $\mathcal{M}_{0,3}([S^2], i)$ can be understood as the set of all tuples $(S^2, i, (0, 1, \infty), \varphi)$ where $\varphi : (S^2, i) \to (S^2, i)$ ranges over all holomorphic maps of degree 1, i.e. biholomorphic transformations of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. The latter are the fractional linear transformations,

$$\varphi : S^2 \to S^2 : z \mapsto \frac{az + b}{cz + d}, \qquad \text{for} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}),$$

also known as *Möbius transformations*, and since two such matrices produce the same transformation on S^2 if and only if they differ by a sign, we have thus identified $\mathcal{M}_{0,3}([S^2], i)$ with the *projective* special linear group

$$\mathcal{M}_{0,3}([S^2], i) \cong \mathrm{PSL}(2, \mathbb{C}) := \mathrm{SL}(2, \mathbb{C})/\{\pm \mathbb{1}\}.$$

Notice that since S^2 has dimension $2n = 2$ and $c_1([S^2]) := \langle c_1(TS^2, i), [S^2] \rangle = \chi(S^2) = 2$, the index formula from Lecture 8 gives

$$\text{vir-dim}\, \mathcal{M}_{0,3}([S^2], i) = (n - 3)\chi(S^2) + 2c_1([S^2]) + 2(3) = (-2)2 + 2(2) + 6 = 6,$$

and this matches the dimension of the Lie group $\mathrm{PSL}(2, \mathbb{C})$.

(a) Prove by direct inspection of the linearized Cauchy-Riemann operator that every element of $\mathcal{M}_{0,3}([S^2], i)$ is Fredholm regular.
Hint: One cannot argue that i is a generic almost complex structure on S^2, so the results of Lecture 9 are of no help to you here. But the operator in this case is on a line bundle, and we proved something useful in Lecture 6 about Cauchy-Riemann type operators on line bundles over the sphere.

(b) Since any positive area form on S^2 defines a symplectic form taming i, the space $\mathcal{M}_{0,3}([S^2], i)$ is subject to Gromov's compactness theorem. Describe $\overline{\mathcal{M}}_{0,3}([S^2], i)$ concretely. What is its topology? What can you say about subsequences of arbitrary sequences of Möbius transformations $\varphi \in \mathrm{PSL}(2, \mathbb{C})$?

Agenda for the Übung (19.12.2018). This week's Übung will be only one hour (Wednesday at 3pm sharp in 1.315) since I have to run a meeting in the same room at 4:00. We will definitely discuss Exercise 9.1 on the localization of genericity conditions, and will then discuss how to use Gromov's compactness theory to extend the transversality results in Lecture 8 to produce a comeager (rather than just dense) set of regular almost complex structures in $\mathcal{J}(M, \omega)$. If time permits, we will also talk about Exercise 10.2.

11. Gromov's nonsqueezing theorem (15.01.2019)

Topics and reading. The proof I gave for the nonsqueezing theorem is explained in detail in [Wena, Chapter 5], though with one slight difference in presentation: the proof in lecture used Gromov's compactness theorem, whereas [Wena] avoids citing the general compactness theorem and instead gives a direct compactness proof for the situation at hand, using a special case of the same "bubbling off" argument that appears in standard proofs of Gromov compactness.

- The symplectic embedding question and volume obstruction
- Statement of the nonsqueezing theorem
- Reduction to **Theorem**: If there is a symplectic embedding $(\overline{B}_r^{2n}, \omega_{\mathrm{st}}) \hookrightarrow (S^2 \times M, \sigma \oplus \omega)$ for some area form σ on S^2 and a closed symplectic $(2n-2)$-manifold (M, ω) with $\pi_2(M) = 0$, then $\pi r^2 \leqslant \int_{S^2} \sigma$.
- Monotonicity lemma: For nonconstant proper holomorphic maps $u : (\Sigma, j) \to (B_{r_0}^{2n}, i)$ passing through 0, $\int_{u^{-1}(B_r^{2n})} u^* \omega_{\mathrm{st}} \geqslant \pi r^2$ for all $r \in (0, r_0)$.

- Lemma 2: For all compatible J on $(X, \Omega) := (S^2 \times M, \sigma \oplus \omega_M)$ with M closed and $\pi_2(M) = 0$, every point in X is in the image of some J-holomorphic sphere homologous to $A_0 := [S^2 \times \{\text{const}\}] \in H_2(S^2 \times M)$.
- Proof of monotonicity lemma, part 1: The function $F(r) := \frac{1}{r^2} \int_{u^{-1}(B_r^{2n})} u^*\omega_{\text{st}}$ satisfies $\lim_{r \to 0} F(r) = k\pi$ for some $k \in \mathbb{N}$. (Proof via Taylor series of u at a point $z_0 \in \Sigma$ with $u(z_0) = 0$.) At the end of the course we will use some contact geometry to show that $F(r)$ is also nondecreasing in r. (From a different perspective, this is a standard result in the theory of minimal surfaces.)
- Proof of Lemma 2: Choose $J_0 = i \oplus J_M$, so that the evaluation map $\text{ev}_{J_0} : \mathcal{M}_{0,1}(A_0, J_0) \to S^2 \times M$ is a diffeomorphism, then for a given (generic) $J \in \mathcal{J}(X, \Omega)$, extend this to a generic smooth family $\{J_\tau \in \mathcal{J}(X, \Omega)\}_{\tau \in [0,1]}$ with $J_1 = J$ and study the parametric moduli space
$$\mathcal{M}(\{J_\tau\}) := \left\{(\tau, u) \;\middle|\; \tau \in [0,1] \text{ and } u \in \mathcal{M}_{0,1}(A_0, J_\tau)\right\}.$$
Idea is to prove $\text{ev}_J : \mathcal{M}_{0,1}(A_0, J) \to S^2 \times M$ is surjective because $\deg_2(\text{ev}_J) \neq 0 \in \mathbb{Z}_2$.
  - Step 0: All $u \in \mathcal{M}_{0,1}(A_0, J_0)$ are Fredholm regular. (This can be proved by explicit examination of the linearized Cauchy-Riemann operator since the curves are so explicit: one can appeal to the fact from Lecture 6 that Cauchy-Riemann operators on line bundles over spheres are always surjective if they have nonnegative index.)
  - Step 1: Since A_0 is primitive, all $u \in \mathcal{M}_{0,1}(A_0, J)$ are somewhere injective, thus they are regular if J is generic, proving $\mathcal{M}_{0,1}(A_0, J)$ is a smooth manifold of dimension equal to $\text{vir-dim}\, \mathcal{M}_{0,1}(A_0, J) = 2n$.
  - Step 2: $\mathcal{M}_{0,1}(A_0, J)$ is also compact. This is an application of Gromov's compactness theorem, using the assumption $\pi_2(M) = 0$ (see Exercise 11.1 below).
  - Step 3: By the same arguments $\mathcal{M}(\{J_\tau\})$ is a compact smooth manifold of dimension $2n+1$ with boundary $\mathcal{M}_{0,1}(A_0, J_0) \amalg \mathcal{M}_{0,1}(A_0, J)$, hence the map $\text{ev} : \mathcal{M}(\{J_\tau\}) \to S^2 \times M$ is a bordism between $\text{ev}_{J_0} : \mathcal{M}_{0,1}(A_0, J_0) \to S^2 \times M$ and $\text{ev}_J : \mathcal{M}_{0,1}(A_0, J) \to S^2 \times M$, proving $\deg_2(\text{ev}_J) = \deg_2(\text{ev}_{J_0})$. The latter is 1 since ev_{J_0} is a diffeomorphism, thus ev_J is surjective.
  - Step 4: This was not mentioned in the lecture, but if J is not generic, one can still use a compactness argument to prove ev_J is surjective, even if $\mathcal{M}_{0,1}(A_0, J)$ is not smooth (in which case $\deg_2(\text{ev}_J)$ is not defined). Just pick a sequence $J_k \to J$ such that all the J_k are generic, find a sequence $u_k \in \mathcal{M}_{0,1}(A_0, J_k)$ passing through any desired point, and repeat step 2 to find a subsequence of u_k converging to an element of $\mathcal{M}_{0,1}(A_0, J)$.

Exercises.

Exercise 11.1. Work out the details of the compactness argument we used in the proof of the nonsqueezing theorem: namely, if (M, ω) is a $(2n-2)$-dimensional closed symplectic manifold with $\pi_2(M) = 0$, σ is an area form on S^2 and $J \in \mathcal{J}(S^2 \times M, \sigma \oplus \omega)$, prove that every sequence in $\mathcal{M}_{0,1}(A_0, J)$ for $A_0 := [S^2 \times \{\text{const}\}] \in H_2(S^2 \times M)$ has a convergent subsequence.
Hint: According to Gromov's compactness theorem, a subsequence must converge to some stable nodal J-holomorphic curve with one marked point and arithmetic genus 0. The latter implies that all components of the nodal curve are also spheres, and moreover, the graph that has the components as vertices and nodes as edges must be a tree, i.e. it cannot have any cycles. The Euler characteristic of this graph is therefore 1. The total homology class must be A_0, but since all maps $S^2 \to M$ are nullhomotopic, this will impose a strong constraint on the homology classes of the individual components, making most of them constant. By stability, the number of marked

points plus nodal points on each constant component must be at least 3. *Given all this information, your task is to prove that there cannot be any nodes.*

12. Gromov-Witten invariants (16.01.2019)

Topics and reading. For a brief introduction to Gromov-Witten invariants from the perspective presented in the lecture, including a more detailed discussion of pseudocycles, see [Wen18, §7.2]. This section also carries out the proof that the evaluation map on the space of simple curves in dimension four is a pseudocycle, but we'll talk more about that (in slightly different contexts) over the next few weeks.

- Basic question of enumerative geometry: given $J \in \mathcal{J}(M, \omega)$, how many (up to parametrization) J-holomorphic curves u with a given genus and homology class exist with $m \geqslant 0$ marked points $\zeta_1, \ldots, \zeta_m$ satisfying the constraint $u(\zeta_i) \in \bar{\alpha}_i$ for some given submanifolds $\bar{\alpha}_1, \ldots, \bar{\alpha}_m \subset M$?
- "Theorem": If suitably interpreted and $\text{vir-dim}\, \mathcal{M}_{g,m}(A, J) + \sum_{i=1}^m \dim \bar{\alpha}_i = 2mn$, the question has a well-defined answer in $\mathbb{Q}$ that is independent of the choice of $J \in \mathcal{J}(M, \omega)$, depends on the submanifolds $\bar{\alpha}_i$ only up to homology, and depends on ω only up to smooth homotopy in the space of symplectic forms (i.e. *symplectic deformation*).
- Fantasy definition, assuming $\overline{\mathcal{M}}_{g,m}(A, J)$ is a closed oriented manifold of the correct dimension:

$$\begin{aligned}\mathrm{GW}^{(M,\omega)}_{g,m,A} : & H^*(M)^{\otimes m} \to \mathbb{Q} : \alpha_1 \otimes \ldots \otimes \alpha_m \mapsto \mathrm{ev}_*[\overline{\mathcal{M}}_{g,m}(A, J)] \cdot [\bar{\alpha}_1 \times \ldots \times \bar{\alpha}_m] \\ &= \left\langle \mathrm{ev}_1^* \alpha_1 \cup \ldots \cup \mathrm{ev}_m^* \alpha_m, [\overline{\mathcal{M}}_{g,m}(A, J)] \right\rangle = \int_{\overline{\mathcal{M}}_{g,m}(A,J)} \mathrm{ev}_1^* \alpha_1 \cup \ldots \mathrm{ev}_m^* \alpha_m.\end{aligned}$$

  Here "$\cdot$" denotes the homological intersection number, $\alpha_i \in H^*(M)$ is the class Poincaré dual to the submanifold $\bar{\alpha}_i \subset M$, the evaluation map is denoted by

$$\begin{aligned}\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_m) : \overline{\mathcal{M}}_{g,m}(A, J) &\to M^{\times m} \\ [(\Sigma, j, (\zeta_1, \ldots, \zeta_m), \Delta, u)] &\mapsto (u(\zeta_1), \ldots, (u(\zeta_m)),\end{aligned}$$

  and $\mathrm{GW}^{(M,\omega)}_{g,m,A}(\alpha_1, \ldots, \alpha_m) := \mathrm{GW}^{(M,\omega)}_{g,m,A}(\alpha_1 \otimes \ldots \otimes \alpha_m)$ is defined to be 0 unless the dimensional conditions are correct for the intersection number to make sense, which means $\sum_i \deg(\alpha_i) = \text{vir-dim}\, \mathcal{M}_{g,m}(A, J)$.
- Invariance under symplectic deformation: fantasy proof via parametric moduli space and bordism
- Definition: (M, ω) is **symplectically uniruled** if there exists $m \in \mathbb{N}$, $A \in H_2(M)$ and $\alpha_2, \ldots, \alpha_m \in H^*(M)$ such that

$$\mathrm{GW}^{(M,\omega)}_{0,m,A}\left(\mathrm{PD}[\mathrm{pt}], \alpha_2, \ldots, \alpha_m\right) \neq 0,$$

  where $\mathrm{PD}[\mathrm{pt}] \in H^{2n}(M)$ is the Poincaré dual to the homology class of a point. This implies that for all $J \in \mathcal{J}(M, \omega)$, there is a J-holomorphic sphere homologous to A through every point in M.
- Example from previous lecture: if $\pi_2(M) = 0$, then $(S^2 \times M, \sigma \oplus \omega)$ is symplectically uniruled.
- Trouble: $\overline{\mathcal{M}}_{g,m}(A, J)$ is almost never actually a smooth manifold of dimension $d := \text{vir-dim}\, \mathcal{M}_{g,m}(A, J)$, which makes the fundamental class $[\overline{\mathcal{M}}_{g,m}(A, J)] \in H_d(\overline{\mathcal{M}}_{g,m}(A, J)]$ difficult to define.

- Science fiction (but not fantasy), assuming $\mathcal{M}_{g,m}(A,J)$ is always a smooth manifold of the correct dimension: then every curve in $\overline{\mathcal{M}}_{g,m}(A,J)$ with nodes belongs to a smooth "stratum" with dimension $\leqslant \operatorname{vir-dim} \mathcal{M}_{g,m}(A,J) - 2$ (see Exercise 12.1 below).
- Definition of Ω-limit set, d-dimensional pseudocycles and bordism between pseudocycles (see [Wen18, pp. 148–153])
- Intersection number between pseudocycles and proof (in a special case) that it only depends on bordism classes
- Example: In our science-fictional world where $\mathcal{M}_{g,m}(A,J)$ is always smooth with the correct dimension (but not necessarily compact), $\mathrm{ev} : \mathcal{M}_{g,m}(A,J) \to M^{\times m}$ is a pseudocycle.
- Theorem (nonfiction): If $\dim M = 4$ and either $g = 0$ or $\operatorname{vir-dim} \mathcal{M}_{g,0}(A,J) > 0$, then $\mathrm{ev} : \mathcal{M}^*_{g,m}(A,J) \to M^{\times m}$ is a pseudocycle for generic J, where $\mathcal{M}^*_{g,m}(A,J)$ denotes the open set of somewhere injective curves in $\mathcal{M}_{g,m}(A,J)$. In particular,
$$\mathrm{GW}^{(M,\omega)}_{g,m,A}(\alpha_1, \ldots, \alpha_m) := \mathrm{ev} \cdot (\bar{\alpha}_1 \times \ldots \times \bar{\alpha}_m)$$
is then well defined as the intersection number between a pseudocycle and a closed submanifold. (In fact, in this case it is also an integer.)[3]

Exercises.

Exercise 12.1. If $[(\Sigma, j, \Theta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(A,J)$ is a nodal curve, we say that u belongs to a **stratum** in $\overline{\mathcal{M}}_{g,m}(A,J)$ consisting of all curves in the same connected component with u that also have the same nodal configuration. Formally, one can define this as follows: suppose u has connected components $v_i : (S_i, j) \to (M, j)$ for $i = 1, \ldots, N$, so $S = S_1 \amalg \ldots \amalg S_N$. Let $A_i := [v_i] \in H_2(M)$, let $g_i \geqslant 0$ denote the genus of S_i, $m_i \geqslant 0$ the number of the m marked points that lie on S_i, and $n_i \geqslant 0$ the number of nodal points (i.e. individual points that belong to any of the unordered pairs in Δ) on S_i. We can then regard each v_i as an element of $\mathcal{M}_{g_i, m_i+n_i}(A_i, J)$ by treating each of the nodal points as a marked point, but moving each v_i around independently in its respective moduli space will not always produce nodal curves, because we also need to make sure $v_i(z_+) = v_j(z_-)$ for every matching pair $\{z_+, z_-\} \in \Delta$. In other words, $(v_1, \ldots, v_N)$ satisfy condition

$$(\mathrm{ev}(v_1), \ldots, \mathrm{ev}(v_N)) \in Q \subset M^{\times(m_1+n_1)} \times \ldots \times M^{\times(m_N+n_N)}, \tag{12.1}$$

for a submanifold Q determined by this matching condition, e.g. if $m = 0$, and there are only two components v_1, v_2 and one nodal pair $\{z_1, z_2\} \in \Delta$ with $z_1 \in S_1$ and $z_2 \in S_2$, then we are considering two evaluation maps $\mathrm{ev} : \mathcal{M}_{g_i,1}(A_i, J) \to M$ for $i = 1, 2$ with the incidence condition

$$(\mathrm{ev}(v_1), \mathrm{ev}(v_2)) \in \{(x, x) \in M \times M \mid x \in M\} \subset M \times M.$$

We define the stratum of u to be the set of all tuples $(v_1, \ldots, v_N)$ that satisfy this incidence condition, so that they all give rise to nodal curves in $\overline{\mathcal{M}}_{g,m}(A,J)$ by viewing the extra marked points as elemenets in nodal pairs. If the spaces $\mathcal{M}_{g_i, m_i+n_i}(A_i, J)$ are always smooth manifolds of the correct dimension and the intersection of the product of evaluation maps in (12.1) with Q is

[3] There are various reasons why the Gromov-Witten invariants need to be rational numbers instead of integers in more general cases, but they are hard to explain since these are all cases in which the definition of the invariants requires cleverer ideas that we haven't discussed (e.g. Kuranishi structures, or polyfolds). One reason we can point to is that if all J-holomorphic curves were regular, $\mathcal{M}_{g,m}(A,J)$ would still not be a smooth manifold but would have orbifold singularities wherever the curves have nontrivial automorphism group. One can define intersection numbers between orbifolds, but in order to make the definition homotopy invariant, one must divide by the order of the automorphism group wherever an intersection occurs, thus producing a rational number. This is irrelevant in the 4-dimensional settings we are interested in, because one can show that for dimensional reasons, the curves that need to be counted in those settings will never be multiply covered.

transverse, compute what the dimension of the stratum should be. The answer should always be at most $\text{vir-dim}\, \mathcal{M}_{g,m}(A, J) - 2$ unless there are no nodes.

13. Blowups and Lefschetz fibrations (22.01.2019)

Topics and reading. Astonishingly, this week we actually talked about things that are covered in the book [Wen18] I handed out to you at the beginning of the semester. The basic definitions of *symplectic submanifolds* and *symplectic deformation equivalence* are described in Chapter 1, along with the statement of McDuff's theorem on rational and ruled symplectic 4-manifolds. Everything else is covered (in much more detail) in Chapter 3.

- The notion of symplectic submanifolds
- Definition of symplectic deformation equivalence
- The tautological line bundle $\pi : \widetilde{\mathbb{C}}^n \to \mathbb{CP}^{n-1}$ and blowdown map $\beta : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$
- Deifnition of the blowup operation $M \rightsquigarrow \widetilde{M}$ at a point $p \in M$ for complex manifolds, and the exceptional divisor $\mathbb{CP}^{n-1} \cong E \subset \widetilde{M}$
- Proof in case $\dim_{\mathbb{C}} M = 2$ that $\widetilde{M}$ is diffeomorphic to $M \# \overline{\mathbb{CP}}^2$ (where $\overline{\mathbb{CP}}^2$ denotes $\mathbb{CP}^2$ with the reverse of its usual orientation)
- In the case $\dim_{\mathbb{C}} M = 2$, exceptional divisors $S^2 \cong E \subset \widetilde{M}$ have self-intersection number $E \cdot E = -1$
- The symplectic form $\omega_R := \beta^* \omega_{\text{st}} + R^2\, \pi^* \omega_{\text{FS}}$ on $\widetilde{\mathbb{C}}^n$ and symplectomorphism

$$\left(\widetilde{B}_r^{2n} \backslash \mathbb{CP}^{n-1}, \omega_R\right) \xrightarrow{\cong} \left(B^{2n}_{\sqrt{R^2+r^2}} \backslash \overline{B}_R^{2n}, \omega_{\text{st}}\right).$$

- Definition of the symplectic blowup $(M, \omega) \rightsquigarrow (\widetilde{M}, \widetilde{\omega})$ along a symplectically embedded standard ball $(\overline{B}_R^{2n}, \omega_{\text{st}}) \hookrightarrow (M, \omega)$, with exceptional divisor as a symplectic submanifold $(\mathbb{CP}^{n-1}, R^2 \omega_{\text{FS}}) \cong (E, \widetilde{\omega}|_{TE}) \subset (\widetilde{M}, \widetilde{\omega})$
- $(\widetilde{M}, \widetilde{\omega})$ is independent of $R > 0$ and the embedding $(\overline{B}_R^{2n}, \omega_{\text{st}}) \hookrightarrow (M, \omega)$ up to symplectic deformation equivalence
- Lemma (via symplectic neighborhood theorem): Any symplectically embedded 2-sphere $E \subset (M, \omega)$ with $E \cdot E = -1$ (i.e. an **exceptional sphere**) has a neighborhood symplectomorphic to a neighborhood of the zero-section in $(\widetilde{\mathbb{C}}^2, \omega_R)$ for $\pi R^2 = \int_E \omega$. One can thus define the **symplectic blowdown** $(M, \omega) \rightsquigarrow (\breve{M}, \breve{\omega})$ by replacing this neighborhood with a standard ball of radius slightly greater than R.
- Technical lemma: Any $J \in \mathcal{J}(M, \omega)$ such that $J = i$ on $(\overline{B}_R^{2n}, \omega_{\text{st}}) \subset (M, \omega)$ determines (and is determined by) $\widetilde{J} \in \mathcal{J}(\widetilde{M}, \widetilde{\omega})$ with $\widetilde{J} = i$ near the exceptional divisor, such that there is a pseudoholomorphic blowdown map

$$\beta : (\widetilde{M}, \widetilde{J}) \to (M, J).$$

- Definition of a **Lefschetz fibration** $\pi : M \to \Sigma$ for M a closed oriented 4-manifold and Σ a closed oriented surface
- Definition of a **Lefschetz pencil** $\pi : M \backslash B \to \mathbb{CP}^1$ for M a closed oriented 4-manifold and $B \subset M$ discrete
- Example: $\pi : \mathbb{CP}^2 \backslash \{[1:0:0]\} \to \mathbb{CP}^1 : [z_0 : z_1 : z_2] \mapsto [z_1 : z_2]$ is a Lefschetz pencil whose fibers all extend to holomorphically embedded spheres intersecting at one point.
- Big theorem of Donaldson and Gompf (stated without proof): A closed oriented 4-manifold admits a symplectic structure if and only if it admits a Lefschetz pencil.

- Easy direction (also stated without proof): existence and uniqueness up to deformation of symplectic structures compatible with Lefschetz fibrations/pencils (i.e. so that fibers are symplectic), assuming $[\text{fiber}] \in H_2(M)$ is not torsion
- Theorem of McDuff 1990 (our goal for the next few weeks): If (M, ω) is a closed connected symplectic 4-manifold containing a symplectically embedded sphere $S^2 \cong S \subset (M, \omega)$ with $S \cdot S = m \geqslant 0$, then for any choice of m points $B \subset S$, S is a fiber of a symplectic Lefschetz pencil (or fibration in the case $m = 0$) $\pi : M \backslash B \to \Sigma$, and any smooth deformation $\{\omega_\tau\}_{\tau \in [0,1]}$ of the symplectic form can be accompanied by a smooth isotopy of ω_τ-symplectic Lefschetz pencils/fibrations $\pi_\tau : M \backslash B \to \Sigma$. Moreover:
  (1) If $m = 0$, then (M, ω) is a blowup of a **symplectic ruled surface**, meaning a smooth S^2-bundle with symplectic fibers over a closed oriented surface.
  (2) If $m = 1$, then (M, ω) is a blowup of $(\mathbb{CP}^2, c\omega_{\text{FS}})$ for some $c > 0$ (this is called a **rational surface**).
  (3) If $m \geqslant 2$, then one can find another symplectically embedded sphere $S^2 \cong S' \subset (M, \omega)$ with $S' \cdot S' \in \{0, 1\}$ so that one of the first two cases still holds.

Exercises.

Exercise 13.1. Show that a submanifold Σ in a symplectic manifold (M, ω) is symplectic if and only if there exists $J \in \mathcal{J}(M, \omega)$ with $J(T\Sigma) = T\Sigma$.
*Hint: For a point $x \in \Sigma$, if $\omega|_{T_x\Sigma}$ is nondegenerate, then $T_xM = T_x\Sigma \oplus (T_x\Sigma)^{\perp\omega}$, where we define the **symplectic orthogonal complement** of $T_x\Sigma \subset T_xM$ by*

$$(T_x\Sigma)^{\perp\omega} := \left\{ X \in T_xM \;\middle|\; \omega(X, \cdot)|_{T_x\Sigma} = 0 \right\}.$$

Note that for arbitrary (non-symplectic) subspaces $V \subset T_xM$, V and $V^{\perp\omega}$ may generally have nontrivial intersection, though one can show that they are always of complementary dimension.

Agenda for the Übung (25.01.2019). Unless there are other requests, we will discuss two exercises from last week: Exercise 11.1 on the compactness argument for Gromov's nonsqueezing theorem, and Exercise 12.1 on the virtual dimensions of the strata in $\overline{\mathcal{M}}_{g,m}(A, J)$.

14. Dimension four (29.01.2019)

Topics and reading. This week's topics are sketched in reasonable detail in [Wen18, §2.2]. The original proof of the automatic transversality result in [HLS97] is somewhat different from my presentation but also worth looking at; my version follows more closely the approach in [Wen10]. For a more detailed account of intersection theory and the adjunction formula, see [Wenc], and for a complete proof of the Micallef-White theorem (which lies in the background of positivity of intersections and the definition of $\delta(u)$), see [MS12, Appendix E].

- The normal Cauchy-Riemann operator $\mathbf{D}_u^N : W^{k,p}(N_u) \to W^{k-1,p}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$ for an immersed J-holomorphic curve $u : (\Sigma, j) \looparrowright (M, J)$
- $\mathbf{D}_u$ maps tangential part to tangential part
- Proof that u is regular if and only if $\mathbf{D}_u^N$ is surjective
- Theorem (automatic transversality): If $\dim M = 4$, every immersed curve $u : (\Sigma_g, j) \looparrowright (M, J)$ with $\operatorname{ind}(u) > 2g - 2$ is Fredholm regular (J need not be generic).
- Example: Exceptional spheres in blowups are *automatically* regular holomorphic curves (with index 0)
- Definition of the local intersection index $i(u_1, z_1; u_2, z_2) \in \mathbb{Z}$ for two maps $u_i : S_i \to M$ of surfaces S_1, S_2 into a 4-manifold M with an isolated intersection $u_1(z_1) = u_2(z_2)$
- Easy lemma: If u_i are both J-holomorphic and the intersection is transverse, $i(u_1, z_1; u_2, z_2) = +1$ (never -1)

- Harder lemma (positivity of intersections; stated without proof): Isolated intersections of J-holomorphic curves always have local intersection index $\geqslant 1$, with equality iff the intersection is transverse.
- Theorem: For two closed J-holomorphic curves $u_i : (\Sigma_i, j_i) \to (M, J)$ in an almost complex 4-manifold, if their images are not identical, then they have only finitely many intersections and
$$0 \leqslant \left|\left\{(z_1, z_2) \times \Sigma_1 \times \Sigma_2 \;\middle|\; u_1(z_1) = u_2(z_2)\right\}\right| \leqslant u_1 \cdot u_2,$$
where the second inequality is an equality if and only if $u_1 \pitchfork u_2$.
- Example: An exceptional sphere in a blowup is the *only* holomorphic curve in its homology class.
- Lemma: If $u : (\mathbb{D}, i) \to (M, J)$ has a critical point at 0 but is not constant, it admits a C^∞-small perturbation to an immersed J'-holomorphic curve $u' : (\mathbb{D}, i) \to (M, J')$, where J' is a C^∞-small perturbation of J.
- Definition of the singularity count
$$\delta(u) = \frac{1}{2} \sum_{u'(z)=u'(\zeta),\ z \neq \zeta} i(u', z; u', \zeta) \in \mathbb{Z}$$
for $u : (\Sigma, j) \to (M, J)$ a closed somewhere injective curve and u' an immersed perturbation.
- Corollary: $\delta(u) \geqslant 0$ for all simple curves u, with equality if and only if u is embedded.
- Adjunction formula: for simple closed J-holomorphic curves $u : (\Sigma, j) \to (M, J)$,
$$[u] \cdot [u] = 2\delta(u) + c_1([u]) - \chi(\Sigma).$$
- Corollary: $\delta(u)$ depends only on $[u] \in H_2(M)$ and the genus of Σ, hence it is homotopy invariant, i.e. no embedded curve has a simple but non-embedded curve in the same component of the moduli space.

Exercises.

Exercise 14.1. Compute the local intersection index $i(u, 0; v, 0) \in \mathbb{Z}$ for each of the following pairs of holomorphic curves in $\mathbb{C}^2$ with an isolated intersection at the origin:

(a) $u(z) = (z, 0)$ and $v(z) = (z, z^k)$ for some $k \in \mathbb{N}$
(b) $u(z) = (z^3, z^5)$ and $v(z) = (z^4, z^6)$. *Hint: The answer is* 18.

Exercise 14.2. Show that the holomorphic curve $u(z) = (z^3, z^5)$ in $\mathbb{C}^2$ has an immersed holomorphic perturbation u' such that
$$\frac{1}{2} \sum_{u'(z)=u'(\zeta)} i(u', z; u', \zeta) = 10,$$
where the sum is over all pairs $(z, \zeta) \in \mathbb{C} \times \mathbb{C}$ close to $(0, 0)$ with $z \neq \zeta$ but $u'(z) = u'(\zeta)$. Note that this is an integer because for every pair (z, ζ) appearing in the sum there is also (ζ, z). The words "close to $(0, 0)$" mean concretely that for any neighborhood $\mathcal{U} \subset \mathbb{C} \times \mathbb{C}$ of $(0, 0)$, this will be the count of such pairs in $\mathcal{U}$ if the perturbation is sufficiently close to u.

Agenda for the Übung (1.02.2019). We'll discuss Exercises 14.1 and 14.2, plus the theorem of Micallef-White [MW95] that explains why, in some sense, polynomial examples like these tell us everything we need to know about local intersections of pseudoholomorphic curves.

15. Rational and ruled surfaces (5.02.2019)

Topics and reading. The lecture this week was basically a whirlwind survey of the contents of [Wen18, Chapters 4–6]. The results are all due originally to McDuff and appeared first in the paper [McD90].

- Definition: (M^4, ω) is **minimal** if it is not a symplectic blowup (or equivalently, it does not contain an exceptional sphere)
- Theorem 1: Symplectic deformations on closed symplectic 4-manifolds give rise to smooth isotopies of disjoint collections of exceptional spheres (cf. [Wen18, Theorem 5.4])
- Corollary 1: Minimality is invariant under symplectic deformation equivalence
- Corollary 2: Blowing down (M^4, ω) along a maximal disjoint collection of exceptional spheres always produces something minimal (skipped the proof, but see [Wen18, Theorem 5.5])
- Theorem 2: If $S^2 \cong S \subset (M^4, \omega)$ is a symplectically embedded sphere with $S \cdot S = m \geqslant 0$, then for any points $p_1, \ldots, p_m \in S$, S is a fiber of a symplectic Lefschetz pencil or fibration $\pi : M \backslash \{p_1, \ldots, p_m\} \to \Sigma$ such that every singular fiber has exactly one critical point. Moreover, symplectic deformations on M give rise to smooth isotopies of symplectic Lefschetz pencils/fibrations. (See [Wen18, Theorem 6.1])
- Topological operations on Lefschetz fibrations/pencils: (1) blowup at a regular point (creates a new singular fiber with the exceptional sphere as an irreducible component); (2) blowup at a point in the base locus (changes a pencil into a fibration with an exceptional section); (3) reversing operation (1) using Exercise 15.1
- Lemma: If $\pi : M \backslash \{p\} \to \mathbb{CP}^1$ is a symplectic Lefschetz pencil with one base point and fibers of genus 0, then up to symplectic deformation equivalence, $(M, \omega) \cong (\mathbb{CP}^2, \omega_{\mathrm{FS}}) \# N \overline{\mathbb{CP}}^2$ for some $N \geqslant 0$.
- Lemma: If $m \geqslant 2$ in the setting of Theorem 2, then there are singular fibers. (Otherwise blow up $m - 1$ base points and obtain $\mathbb{CP}^2$, which is minimal.)
- Corollary (via Exercise 15.2): In Theorem 2, $m \in \{0, 1\}$ without loss of generality. If $m = 0$, then (M, ω) is a (possibly blown up) symplectic ruled surface, i.e. an S^2-bundle over a closed oriented surface, with symplectic fibers. If $m = 1$, then (M, ω) is symplectic deformation equivalent to $(\mathbb{CP}^2, \omega_{\mathrm{FS}}) \# N \overline{\mathbb{CP}}^2$ for some $N \geqslant 0$ (thus it is a rational surface).
- Moduli spaces $\mathcal{M}^2_{emb}(J) = \mathcal{M}^2_{emb}(J; p_1, \ldots, p_m)$ and $\mathcal{M}^0_{emb}(J) = \mathcal{M}^0_{emb}(J; p_1, \ldots, p_m)$ of embedded spheres with m constrained marked points and $u \cdot u = m$ or $m - 1$
- Main lemma:
  1. All $u \in \mathcal{M}^d_{emb}(J)$ are Fredholm regular for $d = 0, 2$ (and for all J)
  2. $\dim \mathcal{M}^d_{emb}(J) = d$, and every $u \in \mathcal{M}^2_{emb}(J)$ belongs to a unique smooth 2-parameter family of curves whose images foliate an open neighborhood of $M \backslash \{p_1, \ldots, p_m\}$
  3. If $J_k \to J_\infty$ where J_∞ is generic (or belongs to a generic 1-parameter family), then any sequence $u_k \in \mathcal{M}^d_{emb}(J_k)$ has a subsequence convergent to either an element of $\mathcal{M}^d_{emb}(J_\infty)$ or (only if $d = 2$) a nodal curve with two embedded components, each belonging to $\mathcal{M}^0_{emb}(J_\infty; p_{i_1}, \ldots, p_{i_q})$ for some subset $\{p_{i_1}, \ldots, p_{i_q}\} \subset \{p_1, \ldots, p_m\}$, and intersecting each other once transversely.
- Riemann-Hurwitz formula and index relation for multiple covers
- Index relation for nodal curves
- Application of the adjunction formula as u_k degenerates to a nodal curve

Exercises.

Exercise 15.1. Suppose $\pi : M\backslash B \to \Sigma$ is a Lefschetz fibration/pencil with a singular fiber S that has two irreducible components S_1 and S_2 intersecting at a single critical point, such that S_1 has genus zero and does not intersect B. Show that $S_1 \cdot S_1 = -1$, i.e. if the fibration/pencil is compatible with a symplectic structure, then S_1 is an exceptional sphere.

Exercise 15.2. Suppose (M, ω) is minimal and $\pi : M\backslash B \to \mathbb{CP}^1$ is a symplectic Lefschetz pencil with $m \geqslant 1$ base points and fibers of genus zero. Show that any singular fiber with one critical point contains an irreducible component S with $S \cdot S \in \{0, \ldots, m-1\}$.

16. Contact manifolds and symplectic cobordisms (12.02.2019)

Topics and reading. The contents of this lecture are covered in [Wen18, Chapter 8 and §9.1], though we did not have time to discuss Floer homology or any of the analytical issues summarized in §8.3.

- Characteristic line field on a hypersurface $\Sigma^{2n-1} \subset (M^{2n}, \omega)$ and the periodic orbit question for Hamiltonian systems
- Theorem of Rabinowitz-Weinstein 1978: Every star-shaped hypersurface in $(R^{2n}, \omega_{\mathrm{st}})$ admits a periodic orbit.
- Liouville vector fields V and their dual 1-forms $\lambda = \iota_V \omega$
- Convex/contact-type hypersurfaces in symplectic manifolds
- Transverse Liouville vector fields and contact forms (see Exercise 16.1)
- The collar neighborhood of a convex hypersurface (see Exercise 16.2)
- Weinstein conjecture: Closed contact-type hypersurfaces always admit periodic orbits.
- Transverse Liouville vector fields are non-unique but belong to a convex (and therefore contractible) space
- Statement of Gray's stability theorem, definition of a **contact structure** $\xi = \ker \alpha$ and **contact manifold** (S, ξ), why ξ is uniquely determined by ω up to isotopy on a contact-type hypersurface
- Definition: symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+), the **concave** and **convex** boundary respectively; symplectic **fillings** ($M_- = \varnothing$) and symplectic **caps** ($M_+ = \varnothing$)[4]
- Example: $(\bar{B}^{2n}, \omega_{\mathrm{st}})$ is a filling of $(S^{2n-1}, \xi_{\mathrm{st}})$ (definition of ξ_{st}). So is any star-shaped domain; in fact these fillings are symplectically deformation equivalent.
- For any closed symplectic $2n$-manifold (W, ω) and an open Darboux ball $(B_r^{2n}, \omega_{\mathrm{st}}) \subset (W, \omega)$, $(W\backslash B_r^{2n}, \omega)$ is a cap for $(S^{2n-1}, \xi_{\mathrm{st}})$, i.e. there is no meaningful restriction on the possible symplectic caps of standard spheres.
- Theorem: Up to symplectic deformation and blowup, all fillings of (S^3, ξ_{st}) are the same.
- Symplectization $(\mathbb{R} \times M, d(e^s \alpha))$ of a contact manifold $(M, \xi = \ker \alpha)$, and Reeb vector field R_α
- Reformulation of the Weinstein conjecture in terms of closed contact manifolds and Reeb vector fields
- The space $\mathcal{J}(\alpha)$ of almost complex structures on $(\mathbb{R} \times M, d(e^s \alpha))$ determined by a contact form
- Trivial J-holomorphic cylinders over periodic Reeb orbits

[4] You should be aware that the literature contains slight inconsistencies about which manifold is the convex boundary and which is the concave boundary when talking about a "symplectic cobordisms *from A to B*". The majority of authors use these terms the same way that I do, but not all.

- Lemma: If $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ defined on a punctured Riemann surface $\dot{\Sigma} = \Sigma \backslash \Gamma$ has no removable punctures and finite energy
$$E(u) := \sup \left\{ \int_{\dot{\Sigma}} u^* d(e^{\varphi(s)}\alpha) \;\middle|\; \varphi : \mathbb{R} \to [-1, 1],\ \varphi' > 0 \right\}$$
then u is proper and asymptotically approaches a trivial cylinder over a periodic Reeb orbit at each puncture.
- Extension to completed symplectic cobordisms after adding cylindrical ends
- Sketch of a proof of uniqueness of fillings for (S^3, ξ_{st}) using finite-energy holomorphic planes
- Sketch of a proof of Rabinowitz-Weinstein (i.e. the Weinstein conjecture for (S^3, ξ_{st})) using finite-energy holomorphic planes

Exercises.

Exercise 16.1. For V a Liouville vector field in (M^{2n}, ω) with $\lambda = \iota_V \omega$ and $\Sigma \subset M$ a hypersurface, show that $V \pitchfork \Sigma$ iff $\alpha := \lambda|_{T\Sigma}$ is a contact form, i.e. $\alpha \wedge (d\alpha)^{n-1}$ is a volume form.

Exercise 16.2. Given a closed convex hypersurface $S \subset (M, \omega)$ with transverse Liouville vector field V, consider the embedding
$$\Phi : (-\epsilon, \epsilon) \times S \hookrightarrow M : (s, x) \mapsto \varphi_V^s(x),$$
where φ_V^s denotes the time s flow of V. Show that for $\lambda := \iota_V \omega$ and $\alpha := \lambda|_{TS}$, $\Phi^*\lambda = e^s\alpha$ and thus $\Phi^*\omega = d(e^s\alpha)$. Conclude from this that under the obvious diffeomorphisms between all the hypersurfaces $\Phi(\{s\} \times S) \subset M$ for different values of s, the characteristic line fields are the same. Show moreover that if α is a contact form on S, then $d(e^s\alpha)$ is a symplectic form on $\mathbb{R} \times S$.

Exercise 16.3. We didn't have time for it in class, but here is part 2 of the proof of the monotonicity lemma that I promised you in Lecture 11. We assume $u : (\Sigma, j) \to (B^{2n}_{r_0}, i)$ is a nonconstant and proper holomorphic map whose image contains $0 \in B^{2n}_{r_0} \subset \mathbb{C}^n$, and consider the function
$$F(r) = \frac{1}{r^2} \int_{u^{-1}(B_r^{2n})} u^*\omega_{\mathrm{st}}$$
for $r \in (0, r_0]$. We need to show that this function is nondecreasing. It can be reduced to an easy application of Stokes' theorem after reframing the problem in contact-geometric terms.

(a) Show that in the usual Darboux coordinates $(p_1, q_1, \ldots, p_n, q_n)$ on $\mathbb{R}^{2n} = \mathbb{C}^n$ with $\omega_{\mathrm{st}} = \sum_j dp_j \wedge dq_j$, $V := \frac{1}{2}\sum_j \left(q_j \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial p_j} \right)$ is a Liouville vector field.

(b) Use the Liouville vector field V of part (a) to define a diffeomorphism
$$(-\infty, c) \times S^{2n-1} \xrightarrow{\cong} B^{2n}_{r_0} \backslash \{0\} : (s, x) \mapsto \varphi_V^s(s)$$
for a suitable constant $c \in \mathbb{R}$, so by Exercise 16.2, $\Phi^*\omega_{\mathrm{st}} = d(e^s\alpha)$ where $\alpha := \lambda|_{TS^{2n-1}}$ and $\lambda := \iota_V\omega_{\mathrm{st}}$. Show that $J := \Phi^* i \in \mathcal{J}(\alpha)$.

(c) Since u is nonconstant, unique continuation implies that $u^{-1}(0) \subset \Sigma$ is a discrete set, so define a punctured Riemann surface by $\dot{\Sigma} = \Sigma \backslash u^{-1}(0)$. Now use the diffeomorphism in part (b) to rewrite $u : \dot{\Sigma} \to B^{2n}_{r_0}$ as a J-holomorphic curve in the symplectization of $(S^{2n-1}, \xi = \ker \alpha)$,
$$u = (f, v) : (\dot{\Sigma}, j) \to (\mathbb{R} \times S^{2n-1}, J).$$
Show that for all values of r such that $u : \Sigma \to B^{2n}_{r_0}$ intersects the sphere of radius r transversely, $F(r)$ can be expressed in terms of an integral of $v^*\alpha$ over the 1-dimensional submanifold $u^{-1}(\partial \bar{B}^{2n}_r) \subset \Sigma$.

(d) Show that for any contact manifold $(M, \xi = \ker \alpha)$ and any $J \in \mathcal{J}(\alpha)$, if $u = (f, v) : (\Sigma, j) \to (\mathbb{R} \times M, J)$ is J-holomorphic then $v^* d\alpha \geqslant 0$. Use this and Stokes' theorem to conclude that the function $F(r)$ is nondecreasing.

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