Topology II — Humboldt-Universität zu Berlin

C. Wendl, F. Schmäschke — Winter Semester 2018–19

PROBLEM SET 2
To be discussed: 31.10.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. Consider the categories Short and Long, defined as follows. Objects in Short are short exact sequences of chain complexes $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$, with a morphism from this object to another object $0 \to A'_* \xrightarrow{f'} B'_* \xrightarrow{g'} C'_* \to 0$ defined as a triple of chain maps $A_* \xrightarrow{\alpha} A'_*$, $B_* \xrightarrow{\beta} B'_*$ and $C_* \xrightarrow{\gamma} C'_*$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "f"] \arrow[d, "\alpha"] & B_* \arrow[r, "g"] \arrow[d, "\beta"] & C_* \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "f'"] & B'_* \arrow[r, "g'"] & C_* \arrow[r] & 0
\end{tikzcd}$$

The objects in Long are long exact sequences of $\mathbb{Z}$-graded abelian groups $\ldots \to C_{n+1} \xrightarrow{\delta} A_n \xrightarrow{F} B_n \xrightarrow{G} C_n \xrightarrow{\delta} A_{n-1} \to \ldots$, with morphisms from this to another object $\ldots \to C'_{n+1} \xrightarrow{\delta'} A'_n \xrightarrow{F'} B'_n \xrightarrow{G'} C'_n \xrightarrow{\delta'} A'_{n-1} \to \ldots$ defined as triples of homomorphisms $A_* \xrightarrow{\alpha} A'_*$, $B_* \xrightarrow{\beta} B'_*$ and $C_* \xrightarrow{\gamma} C'_*$ that preserve the $\mathbb{Z}$-gradings and make the following diagram commute:

$$\begin{tikzcd}
\ldots \arrow[r] & C_{n+1} \arrow[r, "\delta"] \arrow[d, "\gamma"] & A_n \arrow[r, "F"] \arrow[d, "\alpha"] & B_n \arrow[r, "G"] \arrow[d, "\beta"] & C_n \arrow[r, "\delta"] \arrow[d, "\gamma"] & A_{n-1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\
\ldots \arrow[r] & C'_{n+1} \arrow[r, "\delta'"] & A'_n \arrow[r, "F'"] & B'_n \arrow[r, "G'"] & C'_n \arrow[r, "\delta'"] & A'_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Recall also the category $\mathsf{Top}_{\mathrm{rel}}$, whose objects are pairs (X, A) of topological spaces X with subsets A, with a morphism $(X, A) \to (Y, B)$ being a continuous map of pairs.

   (a) Show that there is a covariant functor $\mathsf{Top}_{\mathrm{rel}} \to \mathsf{Short}$ assigning to each pair (X, A) its short exact sequence of singular chain complexes.

   (b) Show that there is also a covariant functor $\mathsf{Short} \to \mathsf{Long}$ assigning to each short exact sequence of chain complexes the corresponding long exact sequence of their homology groups. (Note that this can be composed with the functor in part (a) to define a functor $\mathsf{Top}_{\mathrm{rel}} \to \mathsf{Long}$.)

   (c) Let $\mathsf{Top}^h_{\mathrm{rel}}$ and Short^h denote categories with the same objects as in $\mathsf{Top}_{\mathrm{rel}}$ and Short respectively, but with morphisms of $\mathsf{Top}^h_{\mathrm{rel}}$ consisting of homotopy classes of maps of pairs, and morphisms of Short^h consisting of triples of chain homotopy classes of chain maps. Show that the functors in parts (a) and (b) also define functors $\mathsf{Top}^h_{\mathrm{rel}} \to \mathsf{Short}^h$ and $\mathsf{Short}^h \to \mathsf{Long}$, which then compose to define a functor $\mathsf{Top}^h_{\mathrm{rel}} \to \mathsf{Long}$.

2. In lecture we defined isomorphisms $S_* : \widetilde{H}_n(X; G) \to \widetilde{H}_{n+1}(SX; G)$ for every space X, abelian group G and $n \in \mathbb{Z}$, where $SX = (X \times [-1, 1]) \big/ \sim$ is the suspension of X, defined via the equivalence relation with $(x, 1) \sim (y, 1)$ and $(x, -1) \sim (y, -1)$ for all $x, y \in X$.

   (a) Show that for any continuous map $f : X \to Y$, the map $Sf : SX \to SY : [(x, t)] \mapsto [(f(x), t)]$ is well defined and continuous, and moreover, that $S(\mathrm{Id}_X) = \mathrm{Id}_{SX}$ and $S(f \circ g) = Sf \circ Sg$ whenever f and g can be composed. In other words, show that S defines a functor $\mathsf{Top} \to \mathsf{Top}$.

(b) Denote by $\widetilde{H}^S_{n+1} : \mathsf{Top} \to \mathsf{Ab}$ the composition of the functor $S : \mathsf{Top} \to \mathsf{Top}$ in part (a) with the functor $\widetilde{H}_{n+1}(\cdot; G) : \mathsf{Top} \to \mathsf{Ab}$ which sends X to $\widetilde{H}_{n+1}(X; G)$. Show that there exists a natural transformation from $\widetilde{H}_n(\cdot; G)$ to $\widetilde{H}^S_{n+1}$ which associates to each space X the isomorphism $S_* : \widetilde{H}_n(X; G) \to \widetilde{H}_{n+1}(SX; G)$.

3. (a) Given a short exact sequence of abelian groups $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, show that the following conditions are equivalent:
   (i) There exists a homomorphism $\pi : B \to A$ such that $\pi \circ f = \mathbb{1}_A$;
   (ii) There exists a homomorphism $i : C \to B$ such that $g \circ i = \mathbb{1}_C$;
   (iii) There exists an isomorphism $\Phi : B \to A \oplus C$ such that $\Phi \circ f(a) = (a, 0)$ and $g \circ \Phi^{-1}(a, c) = c$.

$$\begin{tikzcd}
 & & B \arrow[d, "\Phi"] \arrow[dr, "g"] & & \\
0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr] & & C \arrow[r] & 0 \\
 & & A \oplus C \arrow[ur] & &
\end{tikzcd}$$

If any of these conditions holds, we say that the sequence **splits**.

(b) Show that if the groups in part (a) are all finite-dimensional vector spaces and the homomorphisms are linear maps, then the sequence always splits.

4. If you haven't encountered tensor products of abelian groups before this semester, then you should definitely work through this problem. (If you have, then much of it may seem obvious to you.)

(a) Show that the map $G \oplus H \to G \otimes H : (g, h) \mapsto g \otimes h$ is bilinear, and deduce from this that for any $g \in G$ and $h \in H$, $0 \otimes h = g \otimes 0 = 0 \in G \otimes H$.

(b) Show that for any bilinear map $\Phi : G \oplus H \to K$ of abelian groups, there exists a unique homomorphism $\Psi : G \otimes H \to K$ such that $\Phi(g, h) = \Psi(g \otimes h)$ for all $(g, h) \in G \oplus H$.

(c) Show that for any abelian group G, the map $G \to G \otimes \mathbb{Z} : g \mapsto g \otimes 1$ is a group isomorphism. Write down its inverse.
Hint: Use part (b) to write down homomorphisms in terms of bilinear maps.

(d) Find a natural isomorphism from $(G \oplus H) \otimes K$ to $(G \otimes K) \oplus (H \otimes K)$.

(e) Given two sets S and T, find a natural isomorphism from $F^{\mathrm{ab}}(S) \otimes F^{\mathrm{ab}}(T)$ to $F^{\mathrm{ab}}(S \times T)$. (Here $F^{\mathrm{ab}}(S) = \bigoplus_{s \in S} \mathbb{Z}$ denotes the free abelian group generated by the set S.)

(f) Let $\mathbb{K}$ be a field, regarded as an abelian group with respect to its addition operation. Show that the abelian group $G \otimes \mathbb{K}$ naturally admits the structure of a vector space over $\mathbb{K}$ such that scalar multiplication takes the form

$$\lambda(g \otimes k) = g \otimes (\lambda k)$$

for every $\lambda, k \in \mathbb{K}$ and $g \in G$, and every group homomorphism $\Phi : G \to H$ determines a unique $\mathbb{K}$-linear map $\Psi : G \otimes \mathbb{K} \to H \otimes \mathbb{K}$ such that $\Psi(g \otimes k) = \Phi(g) \otimes k$ for $g \in G$, $k \in \mathbb{K}$.

(g) For any abelian groups A, B, C, D and homomorphisms $f : A \to B$, $g : C \to D$, show that there exists a homomorphism

$$f \otimes g : A \otimes C \to B \otimes D$$

defined uniquely by the condition $(f \otimes g)(a \otimes c) = f(a) \otimes g(c)$ for all $a \in A$ and $c \in C$.

(h) An element $a \in G$ is said to be **torsion** if $ma = 0$ for some $m \in \mathbb{Z}$. Show that if every element of G is torsion and $\mathbb{K}$ is a field (regarded as an abelian group with respect to addition), then $G \otimes \mathbb{K} = 0$.

5. For a space X and abelian group G with singular chain complex $\ldots \to C_2(X; G) \xrightarrow{\partial_2} C_1(X; G) \xrightarrow{\partial_1} C_0(X; G)$, define $\epsilon_* : C_0(X; G) \to G$ by $\epsilon_*(\sum_i g_i \sigma_i) = \sum_i g_i$ for finite sums with $g_i \in G$ and $\sigma_i : \Delta^0 \to X$. Show that $\epsilon_* \circ \partial_1 = 0$, so that $\ldots \to C_2(X; G) \xrightarrow{\partial_2} C_1(X; G) \xrightarrow{\partial_1} C_0(X; G) \xrightarrow{\epsilon_*} G$ also forms a chain complex, and that the homology of this complex is the reduced homology $\widetilde{H}_*(X; G)$ of X.
Hint: You may want to read Remark 28.17 in the lecture notes to see where this definition of ϵ_* comes from.