PROBLEM SET 4
To be discussed: 14.11.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. Use Mayer-Vietoris sequences to compute $H_*(X;\mathbb{Z})$ and $H_*(X;\mathbb{Z}_2)$, where X is

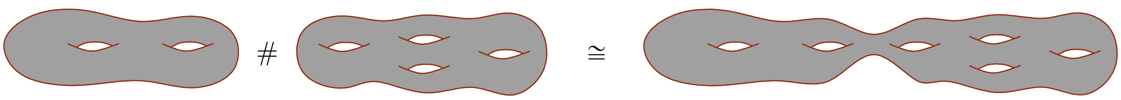
   (a) The projective plane $\mathbb{RP}^2$.

   (b) The Klein bottle.

   Hint: $\mathbb{RP}^2$ is the union of a disk with a Möbius band, and the latter admits a deformation retraction to S^1. The Klein bottle, in turn, is the union of two Möbius bands, also known as $\mathbb{RP}^2 \# \mathbb{RP}^2$.

2. The Klein bottle can also be presented as the mapping torus of $f : S^1 \to S^1 : e^{i\theta} \mapsto e^{-i\theta}$. Use the exact sequence of the mapping torus to verify your answer to Problem 1(b).

3. Recall that given two connected topological n-manifolds X and Y, their **connected sum** $X \# Y$ is defined by deleting an open n-disk $\mathring{\mathbb{D}}^n$ from each of X and Y and then gluing $X \backslash \mathring{\mathbb{D}}^n$ and $Y \backslash \mathring{\mathbb{D}}^n$ together along an identification of their boundary spheres:



   More precisely, we can choose topological embeddings $\iota_X : \mathbb{D}^n \hookrightarrow X$, $\iota_Y : \mathbb{D}^n \hookrightarrow Y$ of the closed unit n-disk $\mathbb{D}^n \subset \mathbb{R}^n$ and then define

$$X \# Y := \left(X \backslash \iota_X(\mathring{\mathbb{D}}^n) \right) \cup_{S^{n-1}} \left(Y \backslash \iota_Y(\mathring{\mathbb{D}}^n) \right),$$

   where the gluing identifies the boundaries of both pieces in the obvious way with $S^{n-1} = \partial \mathbb{D}^n$. There are one or two subtle issues about the extent to which $X \# Y$ is (up to homeomorphism) independent of choices, e.g. in general this need not be true without an extra condition involving orientations, but don't worry about this for now. Last semester (see Problem Set 6 #3) we used the Seifert-van Kampen theorem to show that $\pi_1(X \# Y) \cong \pi_1(X) * \pi_1(Y)$ whenever $n \geqslant 3$. We can now use the Mayer-Vietoris sequence to derive a similar formula for the homology of a connected sum.

   (a) Prove that for any $k = 1, \dots, n-2$ and any coefficient group G, $H_k(X \# Y; G) \cong H_k(X;G) \oplus H_k(Y;G)$.
   Hint: There are two steps, as you first need to derive a relation between $H_k(X;G)$ and $H_k(X \backslash \mathring{\mathbb{D}}^n; G)$, and then see what happens when you glue $X \backslash \mathring{\mathbb{D}}^n$ and $Y \backslash \mathring{\mathbb{D}}^n$ together.

   (b) It turns out that the formula $H_{n-1}(X \# Y; \mathbb{Z}) \cong H_{n-1}(X;\mathbb{Z}) \oplus H_{n-1}(Y;\mathbb{Z})$ also holds if X and Y are both closed orientable n-manifolds with $n \geqslant 2$, and without orientability we still have $H_{n-1}(X \# Y; \mathbb{Z}_2) \cong H_{n-1}(X;\mathbb{Z}_2) \oplus H_{n-1}(Y;\mathbb{Z}_2)$. Prove this under the following additional assumption: $X \backslash \mathring{\mathbb{D}}^n$ and $Y \backslash \mathring{\mathbb{D}}^n$ both admit (possibly oriented) triangulations for which the induced triangulations of $\partial(X \backslash \mathring{\mathbb{D}}^n) = \partial(Y \backslash \mathring{\mathbb{D}}^n) = S^{n-1}$ each define generators of $H_{n-1}(S^{n-1};\mathbb{Z}_2)$ or (in the oriented case) $H_{n-1}(S^{n-1};\mathbb{Z})$.

   (c) Find a counterexample to the formula $H_1(X \# Y;\mathbb{Z}) \cong H_1(X;\mathbb{Z}) \oplus H_1(Y;\mathbb{Z})$ where X and Y are both closed (but not necessarily orientable) 2-manifolds.

4. Recall that for a continuous map $f : X \to X$, one defines the **mapping torus** of f as the space

$$X_f = (X \times [0,1]) \Big/ (x,0) \sim (f(x),1).$$

Assume from now on that f is a homeomorphism. In this case, one can equivalently define X_f as

$$X_f = (X \times \mathbb{R}) \Big/ (x,t) \sim (f(x), t+1)$$

where the equivalence is defined for every $t \in \mathbb{R}$. Take a moment to convince yourself that these two quotients are homeomorphic. The second perspective has the advantage that one can view $\widetilde{X} := X \times \mathbb{R}$ as a covering space for X_f, with the quotient projection defining a covering map $\widetilde{X} \to X_f$ of infinite degree. Writing $S^1 := \mathbb{R}/\mathbb{Z}$, we also see a natural continuous surjective map $\pi : X_f \to S^1 : [(x,t)] \mapsto [t]$, whose **fibers** $\pi^{-1}(t)$ are homeomorphic to X for all $t \in S^1$. We shall denote by $i : X \hookrightarrow X_f$ the inclusion of the fiber $\pi^{-1}([0])$.

In lecture, we proved the existence of a long exact sequence

$$\ldots \longrightarrow h_{k+1}(X_f) \xrightarrow{\Phi} h_k(X) \xrightarrow{\mathbb{1}_* - f_*} h_k(X) \xrightarrow{i_*} h_k(X_f) \xrightarrow{\Phi} h_{k-1}(X) \longrightarrow \ldots$$

for any axiomatic homology theory h_*. The goal of this problem to gain a more concrete picture of the connecting homomorphism $\Phi : H_1(X_f;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ for the special case of singular homology with integer coefficients.

Assume X is path-connected, so there is a natural isomorphism $H_0(X;\mathbb{Z}) = \mathbb{Z}$, and notice that X_f is then also path-connected. Since $H_1(X_f;\mathbb{Z})$ is isomorphic to the abelianization of $\pi_1(X_f, x)$ for any choice of base point $x \in X_f$, we can identify X with $\pi^{-1}([0]) \subset X_f$, fix a base point $x \in X \subset X_f$ and represent any class in $H_1(X_f;\mathbb{Z})$ by a loop $\gamma : [0,1] \to X_f$ with $\gamma(0) = \gamma(1) = x$. Now let $\tilde{\gamma} : [0,1] \to \widetilde{X}$ denote the unique lift of γ to the cover $\widetilde{X} = X \times \mathbb{R}$ such that $\tilde{\gamma}(0) = (x,0)$. Since γ is a loop, it follows that $\tilde{\gamma}(1) = (f^m(x), m)$ for some $m \in \mathbb{Z}$.

(a) Prove that under the natural identification of $H_0(X;\mathbb{Z})$ with $\mathbb{Z}$, the connecting homomorphism $\Phi : H_1(X_f;\mathbb{Z}) \to \mathbb{Z}$ can be chosen[1] such that

$$\Phi([\gamma]) = m,$$

so in particular, $[\gamma] \in \ker \Phi$ if and only if the lift of γ to the cover $\widetilde{X}$ is a loop.

(b) Prove directly from the characterization in part (a) that $\Phi : H_1(X_f;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ is surjective. *Remark: Of course this can also be deduced less directly from the exact sequence.*

[1] There is a bit of freedom allowed in the definition of Φ, e.g. we could replace it with $-\Phi$ and the sequence would still be exact since $\ker \Phi$ and $\operatorname{im} \Phi$ would not change.