PROBLEM SET 7
To be discussed: 5.12.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. Adapt the proof of $H_*^{\mathrm{CW}}(X;G) \cong h_*(X)$ we saw in lecture to prove the relative version of this statement: for any axiomatic homology theory h_* with coefficient group $h_0(\{\mathrm{pt}\}) \cong G$ and any finite-dimensional CW-pair (X, A), i.e. any CW-complex X with a subcomplex $A \subset X$,
$$H_*^{\mathrm{CW}}(X, A; G) \cong h_*(X, A).$$
Explain also how this can be extended to infinite-dimensional CW-pairs when h_* is the singular homology functor $H_*(\,\cdot\,; G)$.
Hint: Start by showing that $C_n^{\mathrm{CW}}(X, A; G)$ is canonically isomorphic to $h_n(X^n \cup A, X^{n-1} \cup A)$, and instead of the long exact sequence of the pair (X^n, X^{n-1}), consider the long exact sequence of the triple $(X^n \cup A, X^{n-1} \cup A, A)$.
Comment: This exercise is a bit lengthy, but it is not fundamentally difficult—every step is simply a minor generalization of something that we discussed in lecture. Working through it is one of the best ways to achieve a deeper understanding of the isomorphism $H_*^{\mathrm{CW}}(X;G) \cong h_*(X)$.

2. The **complex projective n-space** $\mathbb{CP}^n$ is a compact $2n$-manifold defined as the set of all complex lines through the origin in $\mathbb{C}^{n+1}$, or equivalently,
$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\}) / \sim$$
where two points $z, z' \in \mathbb{C}^{n+1}\backslash\{0\}$ are equivalent if and only if $z' = \lambda z$ for some $\lambda \in \mathbb{C}$. It is conventional to write elements of $\mathbb{CP}^n$ in so-called *homogeneous coordinates*, meaning the equivalence class represented by $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ is written as $[z_0 : \ldots : z_n]$. Notice that $\mathbb{CP}^n$ can be partitioned into two disjoint subsets
$$\mathbb{C}^n \cong \{[1 : z_1 : \ldots : z_n] \in \mathbb{CP}^n\} \quad \text{and} \quad \mathbb{CP}^{n-1} \cong \{[0 : z_1 : \ldots : z_n] \in \mathbb{CP}^n\}.$$
   (a) Show that the partition $\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}$ gives rise to a cell decomposition of $\mathbb{CP}^n$ with one $2k$-cell for every $k = 0, \ldots, n$.
   (b) Compute $H_*(\mathbb{CP}^n; G)$ for an arbitrary coefficient group G. *Hint: This is easy.*

3. Each of the following spaces can be defined as a direct limit in terms of the natural inclusions $\mathbb{F}^m \hookrightarrow \mathbb{F}^n$ for $n \geqslant m$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and we identify $\mathbb{F}^m$ with the subspace $\mathbb{F}^m \oplus \{0\} \subset \mathbb{F}^n$. In particular, $\mathbb{R}^{m+1} \hookrightarrow \mathbb{R}^{n+1}$ gives rise to inclusions $S^m \hookrightarrow S^n$ and $\mathbb{RP}^m \hookrightarrow \mathbb{RP}^n$, and the complex version gives $\mathbb{CP}^m \hookrightarrow \mathbb{CP}^n$. Use cell decompositions to compute the homology with integer coefficients for each space:
   (a) $S^\infty = \varinjlim \{S^n\}_{n\in\mathbb{N}}$
   (b) $\mathbb{RP}^\infty = \varinjlim \{\mathbb{RP}^n\}_{n\in\mathbb{N}}$
   (c) $\mathbb{CP}^\infty = \varinjlim \{\mathbb{CP}^n\}_{n\in\mathbb{N}}$

4. Suppose $\{X_\alpha, \varphi_{\beta\alpha}\}$ is a direct system in a category $\mathscr{C}$ over a directed set $(I, \prec)$, with the property that for some $\alpha_0 \in I$, $\varphi_{\gamma\beta} \in \mathrm{Mor}(X_\beta, X_\gamma)$ is an isomorphism for every $\beta, \gamma \in I$ with $\beta \succ \alpha_0$ and $\gamma \succ \alpha_0$. For each $\alpha \in I$, choose $\gamma \in I$ such that $\gamma \succ \alpha$ and $\gamma \succ \alpha_0$, and define
$$\varphi_\alpha := \varphi_{\gamma\alpha_0}^{-1} \circ \varphi_{\gamma\alpha} \in \mathrm{Mor}(X_\alpha, X_{\alpha_0}).$$

(a) Prove that the morphism φ_α does not depend on the choice of the element $\gamma \in I$.

(b) Prove that $\{X_{\alpha_0}, \varphi_\alpha\}$ is a target of the system.

(c) Prove that $\{X_{\alpha_0}, \varphi_\alpha\}$ also satisfies the universal property in the definition of a direct limit, hence $X_{\alpha_0} = \varinjlim\{X_\alpha\}$.

5. Suppose $\{X_\alpha, \varphi_{\beta\alpha}\}$ is a direct system of topological spaces such that each X_α is a subspace of some fixed topological space X, $\beta > \alpha$ if and only if $X_\alpha \subset X_\beta$, and the maps $\varphi_{\beta\alpha} : X_\alpha \to X_\beta$ in this case are the natural inclusions. Identify $\varinjlim\{X_\alpha\}$ with $\coprod_\alpha X_\alpha \big/ \sim$, using the equivalence relation

$$X_\alpha \ni x \sim y \in X_\beta \qquad \Leftrightarrow \qquad \varphi_{\gamma\alpha}(x) = \varphi_{\gamma\beta}(y) \text{ for some } \gamma \in I \text{ with } \gamma > \alpha,\ \gamma > \beta.$$

The disjoint union of the inclusions $X_\alpha \hookrightarrow \bigcup_{\beta \in I} X_\beta$ then descends to the quotient as a bijection

$$\varinjlim\{X_\alpha\} \to \bigcup_{\alpha \in I} X_\alpha,$$

but the following example shows that this need not be a homeomorphism in general. Let $I = (0, 1)$ and consider the family of sets $X_t = \{0\} \cup (t, 1] \subset \mathbb{R}$ for $t \in I$, ordered by inclusion. The union of these sets is $[0, 1]$, but show that the topological space $\varinjlim\{X_t\}$ is not connected.

6. Let $\{X_\alpha\}_{\alpha \in I}$ denote the family of all countable subsets of S^1, ordered by inclusion as in Problem 5.

(a) Show that this forms a direct system of topological spaces whose direct limit is S^1, with its usual topology.

(b) Show that $H_*(\varinjlim\{X_\alpha\}; \mathbb{Z}) \not\cong \varinjlim\{H_*(X_\alpha; \mathbb{Z})\}$.