TOPOLOGY II — HUMBOLDT-UNIVERSITÄT ZU BERLIN
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PROBLEM SET 9
To be discussed: 19.12.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. The goal of this problem is to prove the main theorem stated in lecture about how to compute Tor (i.e. Theorem 41.2 in the notes). In fact, at no extra cost we can prove the natural generalization of this theorem for modules over an arbitrary commutative ring R with unit. Recall that for two R-modules A and G, the R-module $\operatorname{Tor}^R(A, G)$ is defined by

$$\operatorname{Tor}^R(A, G) = H_1(F_* \otimes_R G),$$

where (F_*, f_*) is any projective resolution of A, so that $(F_* \otimes_R G, f_* \otimes \mathbb{1})$ is understood to be a chain complex of R-modules.[1]

   (a) If A is a free R-module, construct a projective resolution (F_*, f_*) with $F_n = 0$ for all $n \geqslant 1$, and conclude from this that $\operatorname{Tor}^R(A, G) = 0$ for every R-module G.

   (b) If (F_*, f_*) and (F'_*, f'_*) are projective resolutions of A and B respectively, construct a projective resolution of $A \oplus B$ using the modules $F_n \oplus F'_n$, and conclude that $\operatorname{Tor}^R(A \oplus B, G) \cong \operatorname{Tor}^R(A, G) \oplus \operatorname{Tor}^R(B, G)$ for every R-module G.

   (c) Suppose $k \in \mathbb{N}$ has the property that no nonzero element $x \in R$ satisfies $kx = 0$. Construct a projective resolution of the quotient module R/kR with $F_1 = F_0 = R$ and $F_n = 0$ for all $n \geqslant 2$, and conclude from this that for every R-module G, $\operatorname{Tor}^R(R/kR, G)$ is isomorphic to the kernel of the map $G \xrightarrow{\cdot k} G$.

   (d) Prove that whenever G is a free R-module, $\operatorname{Tor}^R(A, G) = 0$ for every R-module A.
   Hint: If G is isomorphic to a direct sum of copies of R, what does that mean for the complex $F_* \otimes_R G$?

2. (a) Prove that for any space X with finitely-many path-components[2] and any abelian group G, $H_1(X; G) \cong H_1(X) \otimes G$. *Hint: $H_0(X; \mathbb{Z})$ is always free.*

   (b) Prove that if X is path-connected and has finite fundamental group, then $H_1(X; \mathbb{Q}) = 0$.

3. Using product CW-complexes, describe a cell decomposition of the torus $\mathbb{T}^n$ for every $n \in \mathbb{N}$ such that the cellular boundary map vanishes.[3] Use this to prove that for any axiomatic homology theory h_* with coefficient group G,

$$h_k(\mathbb{T}^n) \cong G^{\binom{n}{k}}$$

for all $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n$.

4. As in Problem 3, describe a cell decomposition of $\Sigma_g \times S^1$ for which the cellular boundary map vanishes. One can use this to compute $H_*(\Sigma_g \times S^1)$, but I would like a more concrete description of $H_2(\Sigma_g \times S^1)$

[1] Recall also that the R-module tensor product $A \otimes_R B$ is defined to satisfy $(ra) \otimes b = r(a \otimes b) = a \otimes (rb)$ for all $r \in R$, $a \in A$ and $b \in B$, in addition to $(a + a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$. For the precise definition of $A \otimes_R B$, see the end of Lecture 42 in the notes.

[2] The assumption of finitely-many path-components can be lifted at the cost of proving a few more properties of the Tor functor than we have proved.

[3] In both this and Problem 4, it is possible to apply the Künneth formula, but not necessary, due to the fact that the cellular boundary map vanishes.

in particular, meaning the following: show that $H_2(\Sigma_g \times S^1)$ is a free abelian group generated by $2g+1$ submanifolds of the form

$$\gamma_1 \times S^1, \dots, \gamma_{2g} \times S^1 \quad \text{and} \quad \Sigma_g \times \{\text{const}\} \subset \Sigma_g \times S^1,$$

where $\gamma_1, \dots, \gamma_{2g}$ are closed 1-dimensional submanifolds of Σ_g. Here, the homology class represented by a closed orientable 2-dimensional submanifold $S \subset \Sigma_g \times S^1$ is understood to mean $i_*[S] \in H_2(\Sigma_g \times S^1)$, with $i : S \hookrightarrow \Sigma_g \times S^1$ denoting the inclusion and $[S] \in H_2(S) \cong \mathbb{Z}$ a chosen generator.
Hint: You can choose your cell decomposition so that each of these submanifolds is presentable as a subcomplex, and the inclusion is then a cellular map.

5. Recall that the topology of a CW-complex X is defined normally as the strongest topology for which the characteristic maps of all cells $\Phi_\alpha : \mathbb{D}^k \to X$ are continuous. Given another CW-complex Y, let Z and Z' denote the set $X \times Y$ with two (potentially) different topologies: we assign to Z the product topology, and to Z' the topology of the product CW-complex induced by the cell decompositions of X and Y.

   (a) Prove that every open set in Z is also an open set in Z', i.e. the identity map $Z' \to Z$ is continuous. *Remark: In general, the identity map $Z' \to Z$ might not be a homeomorphism!*[4]

   (b) Prove that the identity map $Z' \to Z$ is a homeomorphism if X and Y are both compact.

   (c) Prove that a subset $K \subset Z$ is compact if and only if it is compact in Z', and the two subspace topologies induced by Z and Z' on K are the same. Deduce from this that Z and Z' have the same singular homology and cohomology groups.

6. This problem is intended to elucidate in differential-geometric terms the intuitive reason behind the formula $\partial(e^k_\alpha \times e^\ell_\beta) = \partial e^k_\alpha \times e^\ell_\beta + (-1)^k e^k_\alpha \times \partial e^\ell_\beta$ for the boundary map on product CW-complexes.[5]

   Recall first that an **orientation** of a real n-dimensional vector space V means an equivalence class of bases, where two bases are equivalent if they are connected to each other by a continuous family of bases. The fact that the group $\mathrm{GL}(n, \mathbb{R})$ has two connected components (determined by whether the determinant is positive or negative) means that every real vector space of dimension $n > 0$ has exactly two choices of orientation.[6] On an oriented vector space, we call a basis **positive** whenever it belongs to the equivalence class determined by the orientation. A linear isomorphism $V \to W$ between two oriented vector spaces is called **orientation preserving** if it maps positive bases to positive bases, and is otherwise **orientation reversing**.

   A smooth n-manifold M has a **tangent space** T_xM at every point x, which is an n-dimensional vector space. If you haven't seen this notion in differential geometry, then you should just picture M as a regular level-set $f^{-1}(0) \subset \mathbb{R}^k$ of some smooth function $f : \mathbb{R}^k \to \mathbb{R}^{k-n}$ for some $k \in \mathbb{N}$; a famous theorem of Whitney says that every smooth n-manifold can be described in this way if $k \geqslant 2n$. The tangent space T_xM at each point $x \in M$ is then the n-dimensional linear subspace $\ker df(x) \subset \mathbb{R}^k$. With this notion understood, an **orientation of** M means a choice of orientation for every tangent space T_xM such that the orientations vary continuously with x, i.e. every point $x_0 \in M$ has a neighborhood $\mathcal{U} \subset M$ admitting a continuous family of bases $\{(v_1(x), \dots, v_n(x))\}_{x \in \mathcal{U}}$ of the tangent spaces T_xM such that all of them are positive. If M and N are smooth manifolds of the same dimension, then any smooth map $f : M \to N$ has a derivative $df(x) : T_xM \to T_{f(x)}N$ at every point $x \in M$, and we call f an **immersion** if $df(x)$ is an isomorphism for every $x \in M$. If M and N are both oriented, then an immersion $f : M \to N$ is called **orientation preserving/reversing** if $df(x) : T_xM \to T_{f(x)}N$ is orientation preserving/reversing for every $x \in M$.

   (a) Convince yourself that S^2 admits an orientation (i.e. it is **orientable**), but $\mathbb{RP}^2$ and the Klein bottle do not.

[4] This is easily said, but writing down actual counterexamples is surprisingly difficult, e.g. it turns out that they must involve uncountable many cells. For more on such bizarre issues, see `https://arxiv.org/abs/1710.05296`.

[5] For a direct proof of the formula itself, see Proposition 3B.1 on page 269 of Hatcher.

[6] Dimension zero must always be treated as a special case in orientation discussions. For this informal discussion we make our lives easier by assuming all dimensions are positive.

If V and W are both oriented vector spaces, we define the **product orientation** of $V \oplus W$ to be the one such that if $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ are positive bases of V and W respectively, then $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is a positive basis of $V \oplus W$. This notion carries over immediately to a product of manifolds M and N since for each $(x, y) \in M \times N$, $T_{(x,y)}(M \times N)$ can be naturally identified with $T_x M \oplus T_y N$, hence orientations of M and N give rise to a product orientation of $M \times N$.

(b) Show that if M and N are oriented manifolds of dimensions m and n respectively, then for the natural product orientations, the map $M \times N \to N \times M : (x, y) \mapsto (y, x)$ is orientation preserving if either m or n is even, and orientation reversing if both m and n are odd.

If M is an n-manifold with boundary, then its boundary ∂M is naturally an $(n-1)$-manifold, and for each $x \in \partial M$, the tangent space $T_x(\partial M)$ is naturally a codimension 1 linear subspace of $T_x M$. The set $T_x M \backslash T_x(\partial M)$ thus has two connected components, characterized as the tangent vectors in $T_x M$ that point "outward" or "inward" with respect to the boundary. Now if M has an orientation, this induces on ∂M the so-called **boundary orientation**, defined such that for any choice of *outward* pointing vector $\nu \in T_x M$, a basis $(X_1, \ldots, X_{n-1})$ of $T_x(\partial M)$ is positive (with respect to the orientation of ∂M) if and only if the basis $(\nu, X_1, \ldots, X_{n-1})$ of $T_x M$ is positive with respect to the orientation of M. Take a moment to convince yourself that this notion is well defined.

The simplest example is also the most relevant for our discussion of cell complexes: the closed n-disk $\mathbb{D}^n$ is a compact n-dimensional smooth manifold with boundary $\partial \mathbb{D}^n = S^{n-1}$. Since all the tangent spaces to $\mathbb{D}^n$ are canonically isomorphic to $\mathbb{R}^n$, $\mathbb{D}^n$ has a canonical orientation, and this determines a canonical orientation for S^{n-1}.

Finally, consider a product $M \times N$ of two smooth manifolds with boundary, with dimensions m and n respectively. This is a slightly more general object called a "smooth manifold with boundary and corners"; rather than defining this notion precisely, let us simply agree that in the complement of the "corner" $\partial M \times \partial N$, the object $M \times N$ is a smooth manifold whose boundary $\partial(M \times N)$ is the union of two smooth manifolds $\partial M \times N$ and $M \times \partial N$ of dimension $m + n - 1$. The question is: what orientations should these two pieces of $\partial(M \times N)$ carry?

(c) Assume M and N are both oriented, $M \times N$ is endowed with the resulting product orientation and ∂M and ∂N are each endowed with the boundary orientation. Show that the induced boundary orientation on $\partial(M \times N)$ always matches the product orientation of $\partial M \times N$, and that it matches the product orientation of $M \times \partial N$ if and only if m is even.

Remark: The result of part (c) can be summarized as follows. If M has an orientation and we denote the same manifold with the opposite orientation by $-M$, then for any two oriented manifolds M and N of dimensions m and n respectively,

$$\partial(M \times N) = (\partial M \times N) \cup (-1)^m (M \times \partial N).$$

If you apply this to the case $M = \mathbb{D}^m$ and $N = \mathbb{D}^n$ and consider that the degree of a map $S^k \to S^k$ changes sign if you compose it with an orientation-reversing homeomorphism, you may now be able to imagine the reason for the sign in the cellular boundary formula $\partial(e^k_\alpha \times e^\ell_\beta) = \partial e^k_\alpha \times e^\ell_\beta + (-1)^k e^k_\alpha \times \partial e^\ell_\beta$.