# Problem session 1

Topics to be covered today

- Banach spaces – examples ✓
- Absolute convergence of series in Banach spaces ✓
- Lebesgue Dominated convergence theorem and applications (Differential under integral sign)
- Picard – Lindelöf theorem. (Uniqueness & Existence to IVP of ODE)
  
  ↓ Banach fixed point theorem.

## §1. Banach spaces

$X$ vector space over $\underbrace{\mathbb{R}/\mathbb{C}}_{\text{scalars}}$, norm $\|\cdot\|$

is a Banach space if $(X, \|\cdot\|)$ is complete under the metric induced by $\|\cdot\|$.

$\downarrow$ every Cauchy sequence $(x_n)$ in $X$ converges in $X$.

$d(x,y) := \|x-y\|$ , $x, y \in X$

$(x_n)$ is Cauchy if $\exists\ N$ s.t. $\forall\ m, n \geq N$

$\|x_m - x_n\| \longrightarrow 0$.

Examples 1) $[a,b]$ interval

$C([a,b])$ = space of continuos functions on $[a,b]$.

for $f \in C([a,b])$, $\|f\| = \sup\limits_{x \in [a,b]} (f(x))$

$(C([a,b]), \|\cdot\|)$ is a Banach space.

$(f_n) \longrightarrow f \in C([a,b])$.

2) $l^p$, $1 \leq p < \infty$, sequence spaces.

$\xi = (\xi_1, \xi_2, \dots)$ $\xi_i \in \mathbb{R}$

$$\|\xi\|_p = \left( |\xi_1|^p + |\xi_2|^p + \dots \right)^{1/p}$$

$$l^p = \{ \xi \mid \|\xi\|_p < \infty \}$$

Triangle inequality : $\|\xi + x\|_p \leq \|\xi\|_p + \|x\|_p$

Minkowski inequality

$$\left( \sum_{j=1}^{\infty} |\xi_j + x_j|^p \right)^{1/p} \leq \left( \sum_{k=1}^{\infty} |\xi_k|^p \right)^{1/p} + \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

Hölder's inequality

$$\sum_{j=1}^{\infty} |\xi_j x_k| \leq \left( \sum_{k=1}^{\infty} |\xi_k|^p \right)^{1/p} \left( \sum_{m=1}^{\infty} |x_m|^q \right)^{1/q}$$

where $q \in \mathbb{R}$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

<u>Completeness</u> $(\xi_n)$ sequence in $l^p$ Cauchy

$$\begin{array}{lllll}
(\xi_{n1}) & \xi_{11} & \xi_{12} & \xi_{13} & \cdots & \longrightarrow \xi_1 \text{ of } (\xi_n) \\
 & \xi_{21} & \xi_{22} & \xi_{23} & \cdots & \longrightarrow \xi_2 \text{ of } (\xi_n) \\
 & \vdots & & & & \vdots \\
\text{Cauchy} & \xi_{n1} & \xi_{n2} & \xi_{n3} & \cdots \text{ Cauchy} & \longrightarrow \xi_n \text{ of } (\xi_n) \\
 & (\xi_{\infty 1} & \xi_{\infty 2} & \overset{\text{Cauchy}}{\cdots} & \xi_{\infty n}) & \longrightarrow \in l^p
\end{array}$$

$\forall \epsilon > 0, \exists N$ s.t. $\forall m, n \geq N$

$$\|\xi_m - \xi_n\|_p < \epsilon$$

$$\Rightarrow \sum_{j=1}^{\infty} |\xi_{mj} - \xi_{nj}|^p < \epsilon \Rightarrow |\xi_{mj} - \xi_{nj}| < \epsilon \quad \forall j.$$

$\therefore$ $l^p$ is complete.

3) $L^p$ spaces, $1 \leq p < \infty$ Lebesgue spaces

Suppose $(S, \Sigma, \mu)$ be a measure space

$\Sigma$: $\sigma$-algebra on $S$, $\Sigma \subset \mathcal{P}(S)$

i) $S \in \Sigma$

ii) $A \in \Sigma \Rightarrow S \setminus A \in \Sigma$

iii) If $\{A_i\} \in \Sigma \Rightarrow \bigcup A_i \in \Sigma$

$\mu$ measure $\mu : \Sigma \longrightarrow \mathbb{R} \cup \{\pm\infty\}$

$$\mathcal{L}^p = \left\{ f : S \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable} \\ \underbrace{\|f\|_p}_{\text{norm}} = \left(\int |f|^p\right)^{1/p} < \infty \end{array} \right\}$$

$$L^p = \mathcal{L}^p/\mathcal{K} = \left\{ [f] \;\middle|\; \begin{array}{l} f : S \to \mathbb{R}, \text{ measurable} \\ \|f\|_p < \infty \end{array} \right\}$$

$$\mathcal{K} = \left\{ f : S \to \mathbb{R} \;\middle|\; f = 0 \ \begin{array}{l} \mu\text{-almost} \\ \text{everywhere} \end{array} \right\}$$

$$[f] = \{ g \mid f - g = 0 \ \mu \text{ a.e.} \}$$

$$L^\infty = \left\{ f : S \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ measurable} \\ \|f\|_\infty = \inf \{ C \geq 0 \mid |f(x)| \leq C \text{ a.e. } x \} \end{array} \right\}$$

essential supremum

Minkowski's inequality

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Hölder's inequality

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

$L^1$ integrable functions.

4) $S = \Omega \underset{\text{open}}{\subseteq} \mathbb{R}^n$

$$L^p_{\underset{\text{locally}}{\underline{loc}}}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f|_K \in L^p(K) \text{ for all } \underset{\text{compact}}{K \subset \Omega} \right\}$$

$L^1_{loc}(\Omega)$ = locally integrable functions

$\int_K |f|$ - Banach space.

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$f : S \longrightarrow V$ , $V$ - $k$ finite dimensional v.s.

$\{v_1, \ldots, v_k\}$ basis of $V$.

$f = \sum_{j=1}^{k} f_j v_j$ , $f_j : S \longrightarrow \mathbb{R}$

$f(s) = f_1(s) v_1 + f_2(s) v_2 + \cdots + f_k(s) v_k$

$$\int f \, d\mu = \sum_{j=1}^{k} \left( \int f_j \, d\mu \right) v_j \in V$$

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• Bochner integral

$V$ is finite-dim.

$c \|\cdot\|_2 \le \| \, \|_1 \le C \| \, \|_2$ - $\|\cdot\|_1$ , $\|\cdot\|_2$ are equivalent.

- $\int f d\mu$ is well-defined.

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2) Absolute convergence of series in Banach.

$X$ Banach space $\|\cdot\|$

$(x_n)$ sequence in $X$

$S_n = x_1 + x_2 + \dots + x_n$ , $\forall n$.

$(S_n)$ , $S_n = \sum_{j=1}^{n} x_i$ – series

$(S_n)$ converges in $X$ if it converges to some $s \in X$.

$(S_n)$ absolutely converges if $\left( \sum_{j=1}^{n} |x_j| \right) \xrightarrow{n \to \infty} x$

<u>Theorem</u> 1) If $X$ is a Banach space then every absolutely convergent series is convergent.

2) If every absolutely convergent series converges in some normed space $X$, then $X$ must be a Banach space.

<u>Proof</u> 1) $\underline{X}$ is Banach $\sum_{n=1}^{\infty} \|a_n\|$ is convergent

$(a_n)$ $\sum_{n=1}^{\infty} a_n$ is convergent.

Cauchy

$\sum \|a_n\|$ is convergent in $\mathbb{R} \Rightarrow \sum \|a_n\|$ is Cauchy

$\forall \epsilon > 0$, $\exists N$, s.t. $\forall$ $m, n \geq N$

$$\sum_{k=1}^{m} \|a_k\| - \sum_{k=1}^{n} \|a_k\| < \epsilon \quad \text{—— (1)}$$

for the same $N$, $m, n$ , $\sum a_k$

$$\left\| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k \right\| = \left\| \sum_{k=n+1}^{m} a_k \right\|$$

$$\underbrace{\phantom{\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k}}_{< \epsilon} \qquad \leq \sum_{k=n+1}^{m} \| a_k \| \quad (\text{triangle} \leq)$$

$$< \epsilon$$

by (1)

$\therefore$ $(S_n) = \sum_{k=1}^{n} a_k$ is a Cauchy sequence in $X$

$\therefore \longrightarrow X$.

(2) $X$ - norm, v.s. Let's $(x_n)$ be a Cauchy sequence in $X$. $\Longrightarrow$ $\exists N$ s.t.

$\forall n_k \geq N \qquad |x_{n_{k+1}} - x_{n_k}| \leq \frac{1}{2^k}$ — (2)

$(y_k)$ is a subsequence of $(x_n)$

form this sequence

$\underline{y_k = x_{n_{k+1}} - x_{n_k}}$, $\quad y_1 = x_{n_1}$

$y_2 = x_{n_2} - x_{n_1}$

$\vdots$

$\sum y_k$ is absolute convergent

$\hookrightarrow \sum |y_k| < \epsilon \Longrightarrow \sum \left| \frac{1}{2^k} \right| < \epsilon$

$\Longrightarrow \sum y_k$ convergence

$\Rightarrow$ subsequence $(y_k)$ of the Cauchy sequence $(x_n)$ which converges $\Rightarrow$ $\underbrace{(x_n) \to x}$ $\in X$

$\Rightarrow$ $X$ Banach space. $\boxed{\checkmark}$

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## Lebesgue Dominated convergence theorem

$(X, \mu)$ measure space

$(f_n) : X \to \mathbb{R}$ measurable functions $\forall n \in \mathbb{N}$

$f_n(x) \xrightarrow{n \to \infty} f(x)$ , $f : X \to \mathbb{R}$
for a.e. $x$

Suppose $\exists$ $g : X \to \mathbb{R}$ integrable, $g \in L^1(\mu)$

s.t. $|f_n(x)| \leq g(x)$ $\forall n$ . a.e. $x$.

then $f_n \in L^1(\mu)$ , $f \in L^1(\mu)$

and $\displaystyle\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$

Proof.

Fatou's Lemma: Suppose $\{f_n\}$ measurable functions w/ $f_n \geq 0$. If $\displaystyle\lim_{n \to \infty} f_n(x) = f(x)$ a.e. $x$

then $\displaystyle\int f \leq \liminf_{n \to \infty} \int f_n .$

Want. $\int_X |f_n| \, d\mu < \infty$

$|f_n| \leq g$ (given) $\xRightarrow[\text{of integral}]{\text{monotonicity}}$ $\int_X |f_n| \, d\mu \leq \int_X g \, d\mu < \infty$

$\Rightarrow$ $f_n \in L^1(\mu)$.

$\because$ $f_n(x) \longrightarrow f(x)$ a.e. $x$

$\Rightarrow$ $|f| \leq g$ a.e. $\Rightarrow \int |f| \, d\mu \leq \int g \, d\mu < \infty$

$\Rightarrow f \in L^1(\mu)$.

Want to prove $\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$

We'll prove $\lim_{n \to \infty} \int |f_n - f| \, d\mu \to 0$.

$|f_n - f| \leq |f_n| + |f| \leq 2g$ a.e. $x$

Consider $h_n = 2g - |f_n - f| \geq 0$ $\forall n$

measurable.

$\left( \liminf_{n \to \infty} \int h_n = \liminf \left( \int 2g - \int |f_n - f| \right) = \int 2g \, d\mu - \limsup \int |f_n - f| \, d\mu \right.$

$\xRightarrow{\text{Fatou's Lemma}}$ $\int \liminf_{n \to \infty} h_n \, d\mu \leq \liminf_{n \to \infty} \int_X h_n \, d\mu$

$\int_X 2g \, d\mu = \int \liminf_{n \to \infty} h_n \, d\mu \leq \int 2g \, d\mu - \limsup_{n \to \infty} \int |f_n - f| \, d\mu$

$\Rightarrow$ $\limsup_{n \to \infty} \int |f_n - f| \, d\mu \leq 0$ —— A

$$0 \leq \liminf_{n\to\infty} \int |f_n - f| \, d\mu \leq \limsup_{n\to\infty} \int |f_n - f| \, d\mu \leq 0 \quad \text{(A)}$$

$$\Longrightarrow \lim_{n\to\infty} \int |f_n - f| \, d\mu = 0 \quad \text{—— (B)}$$

$$\because 0 \leq \left| \lim_{n\to\infty} \int f_n \, d\mu - \int f \, d\mu \right|$$

$$\leq \int |f_n - f| \, d\mu$$

$$\leq 0$$

$$\therefore \lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu.$$

$\boxed{\phantom{x}}$

Counterexample

$[0,1]$ $\quad f_n(x) = \begin{cases} n, & 0 < x \leq \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$

$f_n(x) \longrightarrow 0 \qquad |f_n(x)| \not\leq g(x)$ for any integrable $g$.

$$\lim_{n\to\infty} \int_0^1 f_n(x) \, dx = 1$$

$$\neq.$$

$$\int_0^1 \lim_{n\to\infty} f_n(x) \, dx = 0$$

Differential under the integral sign.