many of the solutions are typed up. soln to Q. 1, 2(a) + 3(a) are at the end.

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Funktionalanalysis
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Problem Set 12

Due: Thursday, 25.02.2021 (24pts total)

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Convention: *$\mathcal{H}$ is a complex Hilbert space.*

Problem 1

Prove:

(a) A self-adjoint operator $A \in \mathscr{L}(\mathcal{H})$ is positive ($A \geqslant 0$) if and only if $\sigma(A) \subset [0, \infty)$.

(b) If $\langle x, Ax \rangle > 0$ for all $x \neq 0 \in \mathcal{H}$, it does not follow that $0 \notin \sigma(A)$.

Problem 2

The *spectral measure* μ_x corresponding to a self-adjoint operator $A \in \mathscr{L}(\mathcal{H})$ and $x \in \mathcal{H}$ is by definition the unique finite regular measure on the Borel sets in $\sigma(A) \subset \mathbb{R}$ such that

$$\langle x, f(A)x \rangle = \int_{\sigma(A)} f \, d\mu_x \quad \text{for all } f \in C(\sigma(A)).$$

(a) Describe μ_x explicitly in the case where $x \in \mathcal{H}$ is an eigenvector of A.

(b) Describe μ_x explicitly in the case where A is compact and $x \in \mathcal{H}$ is arbitrary.

(c) Show that if A has any eigenvalues of multiplicity greater than 1, then $\mathcal{H}$ does not contain any cyclic vector for A.

Answer:
We say $v \in \mathcal{H}$ is cyclic for A if the subspace spanned by the set $\{v, Av, A^2 v, A^3 v, \ldots\} \subset \mathcal{H}$ is dense. If $\lambda \in \sigma(A)$ is an eigenvalue and $E_\lambda \subset \mathcal{H}$ denotes the corresponding eigenspace, then every $v \in \mathcal{H}$ can be written uniquely as $v = v_\lambda + v_\perp$ for $v_\lambda \in E_\lambda$ and $v_\perp \in E_\lambda^\perp$. Then E_λ and $E_\lambda^\perp$ are each A-invariant, so $A^n v = \lambda^n v_\lambda + A^n v_\perp$ with $A^n v_\perp \in E^\perp$ for every integer $n \geqslant 0$. This set cannot be dense if $\dim E_\lambda > 1$ since the orthogonal projection of $A^n v$ to E_λ always lies in the same 1-dimensional subspace.

(d) (∗) Show that in the case $\mathcal{H} = \mathbb{C}^n$, the converse of part (c) also holds: if $\sigma(A)$ contains n distinct eigenvalues, then a cyclic vector $v \in \mathcal{H}$ for A exists. Give an explicit example of v in the case where $A : \mathbb{C}^n \to \mathbb{C}^n$ is diagonal. [5pts]
Hint: The proof of the spectral theorem will tell you where to look for an example.

Answer:
Motivation: in the proof of the spectral theorem, one constructs a measure space (X, μ) and unitary isomorphism $U : \mathcal{H} \to L^2(X, \mu)$ as follows whenever $\mathcal{H}$ admits a cyclic vector x for A. Define $(X, \mu) := (\sigma(A), \mu_x)$, where μ_x is the spectral measure for x, and define a linear map

$$T : C(\sigma(A)) \to \mathcal{H} : f \mapsto f(A)x.$$

Cyclicity implies that the image of this map is dense in $\mathcal{H}$, and the properties of the continuous functional calculus imply $\|Tf\| = \|f\|_{L^2}$ for all $f \in C(\sigma(A))$, so T extends to a unitary isomorphism $L^2(X,\mu) \to \mathcal{H}$, whose inverse we define to be U. One can now check (again using the properties of the continuous functional calculus) that UAU^{-1} is the multiplication operator $T_F : u \mapsto Fu$ for $F(\lambda) := \lambda$. The main point for our present purposes is that since $f(A) = \mathbb{1}$ for the constant function $f(\lambda) = 1$, that constant function is the element of $L^2(X,\mu)$ that U identifies with our cyclic vector x. One can see explicitly that $f \in L^2(X,\mu)$ is cyclic for the multiplication operator T_F, as the finite linear combinations of elements $f, T_F f, T_F^2 f, \ldots$ are precisely the polynomials on $\mathbb{R}$, restricted to $\sigma(A)$. These are dense in $C(\sigma(A))$ since $\sigma(A) \subset \mathbb{R}$ is compact, and they are also dense in $L^2(\sigma(A),\mu_x)$ since $C(\sigma(A))$ is dense in $L^2(\sigma(A),\mu_x)$.

If $A : \mathbb{C}^n \to \mathbb{C}^n$ is diagonal with n distinct eigenvalues, then the reasoning above leads one to expect that $v := (1,\ldots,1)$ is a cyclic vector for A. To prove it, label the coordinates by the corresponding eigenvalues in order to identify $\mathbb{C}^n$ with the space of all functions $u : \sigma(A) \to \mathbb{C}$, on which A acts as $u \mapsto Fu$ for the function $F(\lambda) = \lambda$. That v is cyclic now follows from the fact that every function $\sigma(A) \to \mathbb{C}$ can be approximated arbitrarily well by the restriction of a polynomial function $P : \mathbb{R} \to \mathbb{C}$ to the finite set $\sigma(A)$.

Problem 3
Assume $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, $\mathcal{D} \subset X$ is a subspace, and $X \supset \mathcal{D} \xrightarrow{T} Y$ is a linear operator, possibly unbounded, and not necessarily closed. Prove:

(a) If T is closed, then so is the operator $\mathcal{D} \to Y : x \mapsto Tx + Ax$ for every bounded operator $A \in \mathscr{L}(X,Y)$.

(b) T is closed if and only if the so-called *graph norm* $\|x\|_T := \|x\|_X + \|Tx\|_Y$ on $\mathcal{D}$ is complete.

Now assume $X = Y$ is a complex Banach space.

(c) $(*)$ Show that for every $\lambda \in \mathbb{C}$ such that $\lambda - T : \mathcal{D} \to X$ is bijective, T is closed if and only if the resolvent operator $R_\lambda(T) : X \to X : x \mapsto (\lambda - T)^{-1}x$ is bounded. [4pts][1]

Answer:
By part (a), T is closed if and only if $\lambda - T$ is closed. As the inverse of $\lambda - T$, the graph of $R_\lambda(T)$ is the set of all points $(x,y) \in X \times X$ such that (y,x) is in the graph of $\lambda - T$, and either of these is a closed subspace of $X \times X$ if and only if the other one is. This proves that T is closed if and only if the graph of $R_\lambda(T)$ is closed, so the result now follows from the closed graph theorem.

Next, assume additionally that T is closed. We call $\lambda \in \mathbb{C}$ an *approximate eigenvalue* of T if there exists a sequence $x_n \in \mathcal{D}$ such that $\|x_n\|_X = 1$ and $(\lambda - T)x_n \to 0$, and λ belongs to the *residual spectrum* of T of the image of $\lambda - T : \mathcal{D} \to X$ is not dense. Prove:

(d) If $\lambda \in \sigma(T)$ is not in the residual spectrum of T, then it is an approximate eigenvalue.

Answer:
Since $\lambda - T$ is a closed operator by part (a), we can use part (b) to regard $\lambda - T$ as a bounded linear operator from the Banach space $(\mathcal{D}, \|\cdot\|_{\lambda-T})$ to $(X, \|\cdot\|_X)$. If

[1] This result is the reason why one normally never considers the spectrum of a non-closed operator.

λ is an eigenvalue then it is also an approximate eigenvalue, so assume from now on that $\lambda \in \sigma(T)$ is in neither the residual nor the point spectrum. Then $\lambda - T$ is injective and not surjective but has dense image, so in particular, its image is not closed. Problem 4(a) from the take-home midterm then implies that there cannot be any estimate of the form $\|(\lambda - T)x\|_X \geqslant c\|x\|_{\lambda-T}$, in other words, there is no lower bound on the ratio

$$\frac{\|(\lambda - T)x\|_X}{\|x\|_{\lambda-T}} = \frac{\|(\lambda - T)x\|_X}{\|x\|_X + \|(\lambda - T)x\|_X} \tag{1}$$

for $x \neq 0 \in \mathcal{D}$. Choose any sequence $x_n \in \mathcal{D}$ for which this ratio tends to zero and normalize it so that $\|x_n\|_X = 1$.

(e) Every approximate eigenvalue of T is in $\sigma(T)$.

Answer:
Eigenvalues are obviously in $\sigma(T)$, so assume $\lambda \in \mathbb{C}$ is an approximate eigenvalue but not an eigenvalue. The map $\lambda - T : \mathcal{D} \to X$ is then injective, and the ratio in (1) has no lower bound, thus the same application of Problem 4(a) from the take-home midterm implies that $\lambda - T$ cannot have closed image; in particular it is not surjective.

Problem 4
Let $\mathrm{AC}^2([0,1])$ denote the space of absolutely continuous complex-valued functions $f(t)$ on $[0,1]$ whose derivatives (defined almost everywhere) are in $L^2([0,1])$.[2] Given the domains

$$\begin{aligned}
\mathcal{D}_0 &:= \mathrm{AC}^2([0,1]),\\
\mathcal{D}_1 &:= \left\{ f \in \mathrm{AC}^2([0,1]) \;\middle|\; f(0) = 0 \right\},\\
\mathcal{D}_2 &:= \left\{ f \in \mathrm{AC}^2([0,1]) \;\middle|\; f(0) = f(1) = 0 \right\},
\end{aligned}$$

consider for $j = 0, 1, 2$ the unbounded operators $L^2([0,1]) \supset \mathcal{D}_j \xrightarrow{T_j} L^2([0,1])$ defined by $T_j := i\partial_t = i\frac{d}{dt}$. Prove:

(a) $(*)$ All three domains are dense in $L^2([0,1])$, and all three operators are closed. [6pts]

Answer: All three domains contain $C_0^\infty((0,1))$, which is dense in $L^2([0,1])$.

To see that T_1 and T_2 are closed, suppose $f_j \in \mathcal{D}_1$ is a sequence with $f_j \xrightarrow{L^2} f \in L^2([0,1])$ and $g_j := if_j' \xrightarrow{L^2} g \in L^2([0,1])$. By the FTC and the assumption $f_j(0) = 0$, $f_j(t) = -i\int_0^t g_j(s)\,ds$ for each j and $t \in [0,1]$. Since $[0,1]$ has finite measure, the L^2-convergence of g_j implies L^1-convergence, thus these integrals converge as $j \to \infty$ and we conclude that f_j converges pointwise to the function $t \mapsto -i\int_0^t g(s)\,ds$. Since the L^2-convergence $f_j \to f$ implies that a subsequence converges to f pointwise almost everywhere, it follows that f is almost everywhere equal to the function $f(t) = -i\int_0^t g(s)\,ds$, which is manifestly in $\mathcal{D}_1$ and satisfies $T_1 f = g$, proving that

[2] You may be interested to know that $\mathrm{AC}^2([0,1])$ is equivalent to the Sobolev space $W^{1,2}((0,1))$. This follows mostly from Problems 3–4 in Problem Set 9: Problem 3 implies that every function in $\mathrm{AC}^2([0,1])$ represents an equivalence class in $W^{1,2}((0,1))$, and that equivalence classes of functions in $W^{1,2}((0,1))$ have unique representatives as continuous functions on $(0,1)$ that are absolutely continuous on compact subsets, and whose derivatives almost everywhere match their weak derivatives. Problem 4 (the Sobolev embedding theorem) implies in turn that these continuous functions are also in the Hölder space $C^{0,\frac{1}{2}}((0,1))$, thus they are uniformly continuous on $(0,1)$, and therefore admit continuous extensions over $[0,1]$. One can deduce from the fundamental theorem of calculus that the extensions are also absolutely continuous.

T_1 is closed. If additionally $f_j \in \mathcal{D}_2$, then $f_j(1) = -i\int_0^1 g_j(s)\,ds = 0$ for every j, thus the L^1-convergence implies $f(1) = -i\int_0^1 g(s)\,ds = 0$ as well and thus $f \in \mathcal{D}_2$, proving that T_2 is closed.

The argument for T_0 seems slightly more complicated since the FTC now gives $f_j(t) = f_j(0) - i\int_0^t g_j(s)\,ds$ but there is no guarantee that $f_j(0)$ converges. One possible remedy is to prove that a subsequence of f_j converges uniformly. I can see two ways to do this: the quickest is perhaps to invoke the observation in the footnote that $\mathrm{AC}^2([0,1]) = W^{1,2}((0,1))$, so that the uniform bounds on $\|f_j\|_{L^2}$ and $\|f_j'\|_{L^2}$ amount to a uniform $W^{1,2}$-bound, and the Sobolev embedding theorem thus gives a uniform $C^{0,\frac{1}{2}}$-bound. This implies that the sequence f_j is uniformly bounded and equicontinuous, so Arzelà-Ascoli does the rest. Alternatively, one can directly prove uniform boundedness and a Hölder bound by using the FTC and Hölder's inequality as in Problem Set 9 #4(c); either way, f_j has a uniformly convergent subsequence, whose limit is therefore f. From this subsequence and the L^1-convergence of g_j we now obtain $f(t) = f(0) - i\int_0^t g(s)\,ds$, thus $f \in \mathcal{D}_0$ and $T_0 f = g$.

(b) Every $\lambda \in \mathbb{C}$ is an eigenvalue of T_0, thus $\sigma(T_0) = \mathbb{C}$.

(c) Every $\lambda \in \mathbb{C}$ is in the resolvent set of T_1, and $(\lambda - T_1)^{-1} : L^2([0,1]) \to \mathcal{D}_1$ sends $g \in L^2([0,1])$ to the function $f(t) := i\int_0^t e^{-i\lambda(t-s)} g(s)\,ds$. In particular, $\sigma(T_1) = \varnothing$.[3]

(d) T_2 is symmetric, but not self-adjoint.

Answer:
In fact, $\langle if', g\rangle_{L^2} = \langle f, ig'\rangle_{L^2}$ for all $f \in \mathcal{D}_2$ and $g \in \mathcal{D}_0$, thus T_2 is symmetric, but the domain of T_2^* contains $\mathcal{D}_0$.

(e) Every $\lambda \in \mathbb{C}$ is in the residual spectrum of T_2, hence $\sigma(T_2) = \mathbb{C}$.

Answer:
For any $\lambda \in \mathbb{C}$, pick a nontrivial eigenvector $g_\lambda \in \mathcal{D}_0$ of T_0. Then for all $f \in \mathcal{D}_2$, $\langle(\lambda - T_2)f, g_\lambda\rangle_{L^2} = \langle f, (\lambda - T_0)g_\lambda\rangle_{L^2} = 0$, implying that g_λ is orthogonal to the image of $\lambda - T_2$.

Problem 5 (*)
Fix an L^2-function $P : [0,1] \to \mathbb{R}$ and define $\mathcal{D}$ to be the vector space of C^1-functions $x : [0,1] \to \mathbb{C}$ such that $x(0) = x(1) = 0$ and the derivative $\dot{x}$ belongs to the space $\mathrm{AC}^2([0,1])$ from Problem 4, so every $x \in \mathcal{D}$ has an almost everywhere defined second derivative $\ddot{x} \in L^2([0,1])$.[4] Setting $Tx := \ddot{x} + Px$, show that $L^2([0,1]) \supset \mathcal{D} \xrightarrow{T} L^2([0,1])$ is an unbounded self-adjoint operator. [5pts]
Hint: Interpret the condition defining the domain of T^* in terms of weak derivatives.

Answer:
If $x \in L^2([0,1])$ is in the domain of T^*, it means there exists a $y \in L^2([0,1])$ such that

[3] The invertibility of $\lambda - T_1$ can also be deduced from general principles without writing down an explicit formula. The essential question is: given $g \in L^2([0,1])$ and $\lambda \in \mathbb{C}$, how many absolutely continuous functions $f : [0,1] \to \mathbb{C}$ satisfy the initial value problem $f' = -i(\lambda f - g)$ with $f(0) = 0$? Intuitively, the Picard-Lindelöf theorem suggests that the answer must be exactly one. Strictly speaking, the standard form of that theorem does not apply here since g cannot be assumed continuous, but if we rewrite the ODE as $f'(t) = H(t, f(t))$ for $H(t,x) := -i(\lambda x - g(t))$, then the more important detail is that H is Lipschitz continuous with respect to x; with a little care, the usual proof of Picard-Lindelöf can then be adapted to prove the existence and uniqueness of a solution, which will be absolutely continuous because it arises as the solution to an integral equation.

[4] Similarly to the situation in Problem 4, $\mathcal{D}$ in Problem 5 is equivalent to the Sobolev space $W^{2,2}((0,1))$.

$\langle y, z \rangle_{L^2} = \langle x, \ddot{z} + Pz \rangle_{L^2}$ and thus

$$\langle x, \ddot{z} \rangle_{L^2} = \langle y, z \rangle_{L^2} - \langle x, Pz \rangle_{L^2} = \langle y - Px, z \rangle_{L^2},$$

for all $z \in \mathcal{D}$, where in the last expression we have used the fact that P is real-valued in order to shift it to the other side of the inner product. Restricting to $z \in C_0^\infty((0,1))$, this implies that x has a weak second derivative $\ddot{x} = y - Px$. Since $y - Px \in L^2([0,1]) \subset L^1([0,1])$, there exists an absolutely continuous function $f(t) := \int_0^t [y(s) - P(s)y(s)]\, ds$ on $[0,1]$ with $\dot{f} = y - Px$ almost everywhere, and by Problem Set 9 #3(a), $y - Px$ is then also a weak derivative of $\dot{f}$, implying that $\dot{x} - f$ has vanishing weak derivative. By a theorem proved in lecture, it follows that the first weak derivative $\dot{x}$ of x is equal to the absolutely continuous function f almost everywhere, and after adjusting x on a set of measure zero, we can then assume x is of class C^1 with classical derivative $\dot{x} = f$. Since the latter has $\dot{f} = y - Px \in L^2([0,1])$, it follows that $T^*x := y = \ddot{x} + Px$, the same expression as T.
It remains only to establish $x \in \mathcal{D}$ by proving $x(0) = x(1) = 0$. Using integration by parts (via the FTC for the Lebesgue integral) and the fact that $z(0) = z(1)$ for all $z \in \mathcal{D}$, we find

$$\begin{aligned} \langle T^*x, z \rangle_{L^2} &= \langle \ddot{x}, z \rangle_{L^2} + \langle Px, z \rangle_{L^2} = -\langle \dot{x}, \dot{z} \rangle_{L^2} + \langle Px, z \rangle_{L^2} \\ &= \langle x, \ddot{z} \rangle_{L^2} - x\dot{z}\big|_0^1 + \langle x, Pz \rangle_{L^2} = \langle x, Tz \rangle_{L^2} - x(1)\dot{z}(1) + x(0)\dot{z}(0). \end{aligned}$$

The relation $\langle T^*x, z \rangle_{L^2} = \langle x, Tz \rangle_{L^2}$ will thus hold for every $z \in \mathcal{D}$ if and only if $x(0)\dot{z}(0) = x(1)\dot{z}(1)$ for every $z \in \mathcal{D}$. Since the values of $\dot{z}(0)$ and $\dot{z}(1)$ can be arbitrary for $z \in \mathcal{D}$, it follows indeed that $x(0) = x(1) = 0$.

Problem 6 (*)
A closed (but not necessarily bounded) self-adjoint operator $\mathcal{H} \supset \mathcal{D} \xrightarrow{A} \mathcal{H}$ is called *positive* if $\langle x, Ax \rangle \geqslant 0$ for all $x \in \mathcal{D}$. Prove (without citing the spectral theorem) that under this assumption, $\sigma(A)$ contains no negative real numbers. [4pts]

Answer:
We need to show that for every $\lambda > 0$, $A + \lambda : \mathcal{D} \to \mathcal{H}$ is bijective. For every $x \in \mathcal{H}$, we have

$$\begin{aligned} \|(A+\lambda)x\|^2 &= \langle Ax + \lambda x, Ax + \lambda x \rangle = \|Ax\|^2 + \lambda^2 \|x\|^2 + 2\lambda \langle x, Ax \rangle \\ &\geqslant \min\{1, \lambda^2\} \cdot \left(\|Ax\|^2 + \|x\|^2 \right) = \min\{1, \lambda^2\} \cdot \|x\|_A^2, \end{aligned}$$

where $\|\cdot\|_A$ denotes the graph norm on $\mathcal{D}$. This shows that $A + \lambda$ is injective. Moreover, since A is closed, Problem 3(b) allows us to view $A + \lambda$ as a bounded linear operator from the Banach space $(\mathcal{D}, \|\cdot\|_A)$ to $\mathcal{H}$, and Problem 4(a) on the take-home midterm then implies that it has closed image. To show that the image is also dense, suppose $v \in (\mathrm{im}(A+\lambda))^\perp$, so $\langle Ax + \lambda x, v \rangle = 0$ for all $x \in \mathcal{D}$, meaning $\langle Ax, v \rangle = \langle -\lambda x, v \rangle = \langle x, -\lambda v \rangle$. This implies that v is in the domain of A^* and $A^*v = -\lambda v$. Since $A = A^*$, this means v is an eigenvector of A with negative eigenvalue, which was already shown to be impossible.

Supplement

The following (unstarred) problem servies two purposes: (1) it fills in some gaps in the lectures' coverage of the spectral theorem for bounded normal operators, and (2) it provides a structured review (with mild generalizations) of the proof of the spectral theorem for bounded self-adjoint operators. From that perspective, working through it informally should serve as valuable preparation for the final exam.

Problem 7
The spectral theorem for bounded normal operators (proved in lecture) provides for any normal operator $A \in \mathscr{L}(\mathcal{H})$ a σ-finite measure space (X, μ), unitary isomorphism $U : \mathcal{H} \to L^2(X, \mu)$ and bounded measurable function $F : X \to \mathbb{C}$ such that $UAU^{-1} = T_F : L^2(X, \mu) \to L^2(X, \mu) : u \mapsto Fu$. One easy corollary is that the Borel functional calculus extends to normal operators, i.e. there is a natural linear map

$$\mathscr{B}(\sigma(A)) \to \mathscr{L}(\mathcal{H}) : f \mapsto f(A) := U^{-1} T_{f \circ F} U,$$

where $\mathscr{B}(\sigma(A))$ denotes the algebra of bounded Borel-measurable functions $f : \sigma(A) \to \mathbb{C}$. Show that this map has the following properties:

(a) $(fg)(A) = f(A)g(B)$, $\bar{f}(A) = A^*$, $f(A) = \lambda \mathbb{1}$ for each constant function $f(z) = \lambda$, and $f(A) = A$ for the identity function $f(z) = z$.

Answer:
Follows directly from the definition of $f(A)$ via the spectral representation

(b) For any pointwise convergent sequence $f_n \to f \in \mathscr{B}(\sigma(A))$ satisfying a uniform bound $\sup_{z \in \sigma(A)} |f_n(z)| \leqslant C$ for all n, $f_n(A)x \to f(A)x$ for every $x \in \mathcal{H}$.

Answer:
Dominated convergence theorem

(c) $f \mapsto f(A)$ is the only linear map $\mathscr{B}(\sigma(A)) \to \mathscr{L}(\mathcal{H})$ satisfying both of the properties in parts (a) and (b).
Hint: Since $\sigma(A) \subset \mathbb{C}$ is compact, Weierstrass implies that the polynomial functions in z and $\bar{z}$ are dense in the space of continuous functions $C(\sigma(A)) \subset \mathscr{B}(\sigma(A))$ with the sup-norm. Similarly, $\mathscr{B}(\sigma(A))$ is the smallest class of functions that contains $C(\sigma(A))$ and is closed under the notion of convergence in part (b).

Answer:
For any polynomial $P : \mathbb{C} \to \mathbb{C} : z \mapsto \sum_{k,\ell} a_{k,\ell} z^k \bar{z}^\ell$, part (a) implies $P(A) = \sum_{k,\ell} a_{k,\ell} A^k (A^*)^\ell$, a result that is clearly independent of the choice of spectral representation. Since any continuous function on $\sigma(A)$ is a pointwise limit of polynomial functions that are uniformly bounded on the compact set $\sigma(A)$, it follows via part (b) that $f(A)$ for $f \in C(\sigma(A))$ is independent of the choice of spectral representation. One can now repeat this type of approximation argument to show that $\chi_\Omega(A)$ is independent of choices whenever $\Omega \subset \mathbb{C}$ is open or closed, as χ_Ω is then a pointwise limit of uniformly bounded continuous functions. By a further repetition, one obtains χ_Ω whenever Ω is an F_σ or G_δ, as χ_Ω is then a pointwise limit of characteristic functions of open or closed sets. Continuing in this way, one eventually obtains χ_Ω for every Borel set $\Omega \subset \mathbb{C}$, and linear combinations of these give uniformly bounded sequences converging pointwise to any $f \in \mathscr{B}(\sigma(A))$.

(d) $f(A)$ is normal for every $f \in \mathscr{B}(\sigma(A))$, and it is self-adjoint whenever $f(\sigma(A)) \subset \mathbb{R}$, positive whenever $f(\sigma(A)) \subset [0,\infty)$, and unitary whenever $f(\sigma(A)) \subset S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

(e) $\sigma(f(A))$ is contained in the closure of $f(\sigma(A)) \subset \mathbb{C}$ for all $f \in \mathscr{B}(\sigma(A))$.
Hint: If $\mu \notin \overline{f(\sigma(A))}$, then $g(z) := \frac{1}{f(z)-\mu}$ belongs to $\mathscr{B}(\sigma(A))$. What is $g(A)$?

(f) $\sigma(f(A)) = f(\sigma(A))$ for all $f \in C(\sigma(A))$.
Hint: By part (e), you only need to show $f(\lambda) \in \sigma(f(A))$ for every $\lambda \in \sigma(A)$. Compare the essential ranges of F and $f \circ F$ (cf. Problem Set 11 #2(a)).

Answer:
If suffices to show that given any bounded measurable function $F : X \to \mathbb{C}$ and a continuous function $f : \sigma(A) \to \mathbb{C}$, if $\lambda \in \mathbb{C}$ is in the essential range of F, then $f(\lambda)$ is in the essential range of $f \circ F$. Note that after adjusting F on a set of measure zero we can assume $F(X) \subset \sigma(A)$, so that $f \circ F$ is defined everywhere. Suppose $\lambda \in \mathbb{C}$ and $\epsilon > 0$: then $(f \circ F)^{-1}(B_\epsilon(f(\lambda))) = \{x \in X \mid |f(F(x)) - f(\lambda)| < \epsilon\}$ contains $F^{-1}(B_\delta(\lambda))$ for some $\delta > 0$ since f is continuous. If λ is in the essential range of F, then $F^{-1}(B_\delta(\lambda))$ has positive measure, implying that $(f \circ F)^{-1}(B_\epsilon(f(\lambda)))$ also has positive measure, thus $f(\lambda)$ is in the essential range of $f \circ F$.

(g) $\|f(A)\| = \|f\|_{C^0}$ for every $f \in C(\sigma(A))$.

Answer:
Since $f(A)$ is normal, Problem Set 11 #1(b) implies that $\|f(A)\|$ is the spectral radius of $f(A)$, so by part (f), this is

$$\|f(A)\| = \sup_{\lambda \in \sigma(f(A))} |\lambda| = \sup_{\lambda \in \sigma(A)} |f(\lambda)| = \|f\|_{C^0}.$$

Here are some applications. Prove:

(h) For $A \in \mathscr{L}(\mathcal{H})$ normal and $\lambda \in \mathbb{C}\backslash\sigma(A)$, the resolvent $R_\lambda(A) := (\lambda - A)^{-1}$ satisfies $\frac{1}{\|R_\lambda(A)\|} = \operatorname{dist}(\lambda, \sigma(A))$.
Answer:
Use the continuous function $f(z) = \frac{1}{\lambda - z}$ on $\sigma(A) \subset \mathbb{C}$ and apply part (g) to compute $\|f(A)\|$.

(i) If $A \in \mathscr{L}(\mathcal{H})$ is normal and $f : \mathcal{D} \to \mathbb{C}$ is holomorphic on some disk $\mathcal{D} = \{|z - z_0| < r\} \subset \mathbb{C}$ containing $\sigma(A)$, with Taylor series $f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n$, then $\sum_{n=0}^\infty a_n (A - z_0 \mathbb{1})^n$ converges absolutely to $f(A)$ in $\mathscr{L}(\mathcal{H})$.

Answer:
The sequence of polynomials $P_k(z) := \sum_{n=0}^k a_n (z - z_0)^n$ converges absolutely and uniformly to f on compact subsets of $\mathcal{D}$, one of which contains $\sigma(A)$, thus by part (g), $\|f(A) - P_k(A)\| \to 0$ as $k \to \infty$, where the operators $P_k(A)$ are the partial sums of the series $\sum_{n=0}^\infty a_n (A - z_0 \mathbb{1})^n$. To see that the convergence in $\mathscr{L}(\mathcal{H})$ is also absolute, we can use part (g) with the continuous function $z \mapsto z - z_0$ to estimate $\|A - z_0\| = \sup_{z \in \sigma(A)} |z - z_0| < r$, so the result follows from the fact that $\sum_{n=0}^\infty a_n (z - z_0)^n$ converges absolutely on $\mathcal{D}$.

(j) An operator $U \in \mathscr{L}(\mathcal{H})$ is unitary if and only if $U = e^{iA}$ for some bounded self-adjoint operator $A \in \mathscr{L}(\mathcal{H})$.

Applying part (d) with the function $f(x) = e^{ix}$ proves that $e^{iA} = f(A)$ is unitary whenever A is self-adjoint. Conversely, if U is unitary, then its spectrum is in $S^1 \subset \mathbb{C}$

by Problem Set 11 #1(c), and we can define a self-adjoint operator $A := g(U)$ by choosing $g : S^1 \to [0, 2\pi)$ as the inverse of $f|_{[0,2\pi)}$; note that g is not continuous, but it is in $\mathscr{B}(S^1)$ and thus restricts to a function in $\mathscr{B}(\sigma(U))$. Choose a spectral representation in order to identify U with a multiplication operator $T_F : L^2(X, \mu) \to L^2(X, \mu) : u \mapsto Fu$. This also defines a spectral representation for A, identifying it with the multiplication operator $T_{g \circ F}$, and it follows that $e^{iA} = f(A)$ is identified with $T_{f \circ g \circ F} = T_F$, hence $e^{iA} = U$.

(k) If $T, A \in \mathscr{L}(\mathcal{H})$ commute and A is normal, then T also commutes with A^*.
Hint: Deduce from $AT = TA$ that $e^{\bar{\lambda} A} T e^{-\bar{\lambda} A} = T$ for all $\lambda \in \mathbb{C}$. Then show that $e^{-\lambda A^*} e^{\bar{\lambda} A}$ is unitary and use this to compute a bound on the norm of $g(\lambda) := e^{-\lambda A^*} T e^{\lambda A^*}$ for all $\lambda \in \mathbb{C}$, concluding that $g : \mathbb{C} \to \mathscr{L}(\mathcal{H})$ is a globally bounded holomorphic function, and thus constant. Finally, compute $\frac{d}{dt} e^{-tA^*} T e^{tA^*}\Big|_{t=0}$.

Answer:
By part (i), $e^{\bar{\lambda} A}$ and $e^{-\bar{\lambda} A}$ are limits in $\mathscr{L}(\mathcal{H})$ of sequences of polynomial functions of A; the latter manifestly all commute with T, thus so do $e^{\bar{\lambda} A}$ and $e^{-\bar{\lambda} A}$. Since $e^{\bar{\lambda} z} e^{-\bar{\lambda} z} = 1$ for all $z \in \mathbb{C}$, it follows via part (a) that

$$e^{\bar{\lambda} A} T e^{-\bar{\lambda} A} = e^{\bar{\lambda} A} e^{-\bar{\lambda} A} T = \mathbb{1} T = T. \tag{2}$$

(Note: One could probably also have proved this by writing down the exponentials as power series, but it would have been more painful.) Next observation: $e^{-\lambda \bar{z}} e^{\bar{\lambda} z} = e^{\bar{\lambda} z - \overline{\bar{\lambda} z}} = e^{2i\Im(\bar{\lambda} z)} \in S^1$ for all $z \in \mathbb{C}$, thus by part (d),

$$U := e^{-\lambda A^*} e^{\bar{\lambda} A}$$

is unitary, and its inverse is

$$U^{-1} = U^* = e^{-\bar{\lambda} A} e^{\lambda A^*}.$$

Unitarity implies $\|UTU^{-1}\| = \|T\|$, so it follows in light of (2) that

$$\|T\| = \|UTU^{-1}\| = \|e^{-\lambda A^*} e^{\bar{\lambda} A} T e^{-\bar{\lambda} A} e^{\lambda A^*}\| = \|e^{-\lambda A^*} T e^{\lambda A^*}\|$$

for all $\lambda \in \mathbb{C}$. Since $e^{-\lambda A^*}$ and $e^{\lambda A^*}$ are both globally convergent operator-valued power series in λ, this proves that $f(\lambda) := e^{-\lambda A^*} T e^{\lambda A^*}$ is a globally bounded holomorphic function $\mathbb{C} \to \mathscr{L}(\mathcal{H})$, so by a standard application of the Cauchy integral formula (in the generalized context of $\mathscr{L}(\mathcal{H})$-valued holomorphic functions), $f(\lambda)$ is constant, meaning $e^{-\lambda A^*} T e^{\lambda A^*} = T$ for all $\lambda \in \mathbb{C}$. Finally, we can specialize to $\lambda = t \in \mathbb{R}$ and differentiate the convergent power series with respect to t, obtaining

$$0 = \frac{d}{dt} e^{-tA^*} T e^{tA^*}\Big|_{t=0} = -A^* T + T A^*.$$

(l) In the setting of part (k), T also commutes with $f(A)$ for every $f \in \mathscr{B}(\sigma(A))$.

Answer:
Since T commutes with both A and (by part (k)) A^*, it commutes with $f(A)$ whenever $f : \mathbb{C} \to \mathbb{C}$ is a polynomial function of z and $\bar{z}$. One then extends this to all $f \in \mathscr{B}(\sigma(A))$ using approximation arguments as in part (c).

Finally, we can now establish some improvements to the spectral theorem:

(m) Under what conditions does a normal operator $A \in \mathscr{L}(\mathcal{H})$ admit a spectral representation of the form $U : \mathcal{H} \to L^2(\sigma(A), \mu)$ with $UAU^{-1} = T_F$ for $F(\lambda) = \lambda$, where μ is a finite measure on $\sigma(A) \subset \mathbb{C}$ such that $C(\sigma(A))$ is dense in $L^2(\sigma(A), \mu)$?

Answer:
One needs the existence of an element $x \in \mathcal{H}$ that is *cyclic for A and A^**, meaning that the set $\{A^k(A^*)^\ell x \mid k.\ell \geqslant 0 \text{ integers}\}$ spans a dense subspace of $\mathcal{H}$. Indeed, this holds if and only if the image of the map

$$T : C(\sigma(A)) \to \mathcal{H} : f \mapsto f(A)x$$

is dense. Let μ_x denote the unique regular Borel measure on $\sigma(A) \subset \mathbb{C}$ such that

$$\langle x, f(A)x\rangle = \int_{\sigma(A)} f \, d\mu_x,$$

as provided by the Riesz-Markov theorem since $f \mapsto \langle x, f(A)x\rangle$ is a positive linear functional $C(\sigma(A)) \to \mathbb{C}$. Note that $\mu_x(\sigma(A)) = \int_{\sigma(A)} 1 \, d\mu_x = \langle x, x\rangle < \infty$, and $C(\sigma(A))$ is dense in $L^2(\sigma(A), \mu_x)$. The properties of the functional calculus then imply that T is an isometry with respect to the L^2-norm on $C(\sigma(A))$:

$$\|Tf\|^2 = \langle f(A)x, f(A), x\rangle = \langle x, f(A)^* f(A), x\rangle = \langle x, |f|^2(A)x\rangle = \int_{\sigma(A)} |f|^2 \, d\mu_x = \|f\|^2_{L^2}.$$

It follows via density that T extends uniquely to a unitary isomorphism $L^2(\sigma(A), \mu_x) \to \mathcal{H}$, and we define $U := T^{-1}$. It is now easy to verify via the properties of the functional calculus that $UAU^{-1} = T_F$ for $F(\lambda) = \lambda$.

Conversely, if a spectral representation $U : \mathcal{H} \to L^2(\sigma(A), \mu)$ with $UAU^{-1} = T_F$ for $F(\lambda) = \lambda$ is given and we assume μ to be any finite measure on $\sigma(A)$ for which $C(\sigma(A))$ is dense in $L^2(\sigma(A), \mu)$, then we claim that the vector $x := U^{-1}f \in \mathcal{H}$ given by the constant function $f(\lambda) := 1$ is cyclic for A and A^*. Indeed, the finite linear combinations of all elements of the form $A^k(A^*)^\ell x \in \mathcal{H}$ correspond under the isomorphism $U : \mathcal{H} \to L^2(\sigma(A), \mu)$ to the complex-valued functions on $\sigma(A) \subset \mathbb{C}$ that are restrictions of polynomials in z and $\bar{z}$. These are dense in $C(\sigma(A))$ with respect to the sup-norm, and since the latter is dense in $L^2(\sigma(A), \mu)$, they are also dense in $L^2(\sigma(A), \mu)$, implying that the space spanned by all $A^k(A^*)^\ell x$ is dense in $\mathcal{H}$.

(n) Show that for every normal operator $A \in \mathscr{L}(\mathcal{H})$, $\mathcal{H}$ splits into a direct sum of (perhaps infinitely many) mutually orthogonal A-invariant subspaces $\mathcal{H}_n \subset \mathcal{H}$ on which $A|_{\mathcal{H}_n}$ admits a spectral representation as described in part (m).

Answer:
Let's just consider the case where $\mathcal{H}$ is separable, so that Zorn's lemma is not required. Choose a dense sequence $y_1, y_2, y_3, \ldots \in \mathcal{H}$, set $x_1 := y_1$, and define $\mathcal{H}_1 \subset \mathcal{H}$ to be the closure of the set of all finite linear combinations of vectors of the form $A^k(A^*)^\ell x_1$ for integers $k, \ell \geqslant 0$. This subspace is invariant under both A and A^*, thus so is its orthogonal complement, as $v \in \mathcal{H}_1^\perp$ implies

$$\langle x, Av\rangle = \langle A^*x, v\rangle = 0 \quad \text{and} \quad \langle x, A^*v\rangle = \langle Ax, v\rangle = 0$$

for all $x \in \mathcal{H}_1$. Define x_2' to be the first element in the sequence y_j such that $x_2' \notin \mathcal{H}_1$, and define $x_2 \in \mathcal{H}_1^\perp$ such that $x_2' = x_2 + z$ for some $z \in \mathcal{H}_1$ (this determines x_2 uniquely). Now define $\mathcal{H}_2$ to be the closure of the set of all finite linear combinations of vectors of the form $A^k(A^*)^\ell x_2$, and observe that since $\mathcal{H}_1^\perp$ is preserved by A and A^*, $\mathcal{H}_2$ is orthogonal to $\mathcal{H}_1$. To continue, set x_3' to be the first element in the sequence y_j such that $x_3' \notin \mathcal{H}_1 \oplus \mathcal{H}_2$, and project it orthogonally to $(\mathcal{H}_1 \oplus \mathcal{H}_2)^\perp$ to define x_3, then define $\mathcal{H}_3$ out of $A^k(A^*)^\ell x_3$ as above. Continuing in this way, it may at some point happen that the entire sequence y_j is contained in $\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_k$ for some finite $k \in \mathbb{N}$, in which case the process terminates. If not, then we obtain a countable sequence of mutually orthogonal closed subspaces $\mathcal{H}_n \subset \mathcal{H}$ that each admit a cyclic vector x_n for A and A^*, and since the span of $\{y_1, y_2, y_3, \ldots\}$ is dense, $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$. On each subspace $\mathcal{H}_n$ individually, $A|_{\mathcal{H}_n}$ is a bounded normal operator admitting a cyclic vector for $A|_{\mathcal{H}_n}$ and $(A|_{\mathcal{H}_n})^* = A^*|_{\mathcal{H}_n}$, thus it admits a spectral representation as in part (m).

(o) When does a finite collection of normal operators $A_1, \ldots, A_N \in \mathscr{L}(\mathcal{H})$ admit a simultaneous spectral representation?

Answer:
This is true if and only if the $A_1, \ldots, A_N$ all commute with each other. In one direction this is obvious since different multiplication operators in the same spectral representation always commute. Conversely, if $A_j A_k = A_k A_j$ for all j, k, then part (l) implies $f(A_j)g(A_k) = g(A_k)f(A_j)$ for all $f \in \mathscr{B}(\sigma(A_j))$ and $g \in \mathscr{B}(\sigma(A_k))$. This is true in particular for all characteristic functions. Let $\mathcal{R} \subset \mathscr{B}(\sigma(A_1) \times \ldots \times \sigma(A_N))$ denote the subspace consisting of restrictions to $\sigma(A_1) \times \ldots \times \sigma(A_N)$ of all finite linear combinations of characteristic functions of "rectangles" $\Omega_1 \times \ldots \times \Omega_N \subset \mathbb{C}^N$, where each $\Omega_j \subset \mathbb{C}$ is a set of the form

$$\Omega_j := \left\{ z \in \mathbb{C} \;\middle|\; a \leqslant \Re z < b,\ c \leqslant \Im z < d \right\}$$

for some $a, b, c, d \in \mathbb{R}$. Note that the closure of $\mathcal{R}$ in the sup-norm over $\sigma(A_1) \times \ldots \sigma(A_N)$ contains the space of continuous functions $C(\sigma(A_1) \times \ldots \times \sigma(A_N))$. We can then define $f(A_1, \ldots, A_N) \in \mathscr{L}(\mathcal{H})$ for every $f(z_1, \ldots, z_N) = \sum_j c_j \chi_{\Omega_j^1}(z_1) \ldots \chi_{\Omega_j^N}(z_N)$ by

$$f(A_1, \ldots, A_N) := \sum_j c_j \chi_{\Omega_j^1}(A_1) \ldots \chi_{\Omega_j^N}(A_N),$$

and verify that this definition satisfies the usual $*$-algebra homomorphism properties. Using the observation that every $f \in \mathcal{R}$ can be written as a sum of terms corresponding to disjoint rectangles, one can then show $\|f(A_1, \ldots, A_N)\| = \|f\|_{L^\infty(\sigma(A_1)\times\ldots\times\sigma(A_N))}$. Thus the map $f \mapsto f(A)$ extends uniquely to an isometry defined on the L^∞-closure of $\mathcal{R}$, which includes $C(\sigma(A_1) \times \ldots \times \sigma(A_N))$.

With a functional calculus for continuous functions on $\sigma(A_1) \times \ldots \times \sigma(A_N)$ in place, one can now use the Riesz-Markov theorem to associate to each $x \in \mathcal{H}$ a spectral measure μ_x on $\sigma(A_1) \times \ldots \times \sigma(A_N)$ such that

$$\langle x, f(A_1, \ldots, A_N)x \rangle = \int_{\sigma(A_1)\times\ldots\times\sigma(A_N)} f \, d\mu_x$$

for every $f \in C(\sigma(A_1) \times \ldots \times \sigma(A_N))$. Call x *cyclic* for $A_1, A_1^*, \ldots, A_N, A_N^*$ if the linear combinations of all elements of the form $A_1^{k_1}(A_1^*)^{\ell_1} \ldots A_N^{k_n}(A_N^*)^{\ell_N} x$ for all integers $k_1, \ell_1, \ldots, k_N, \ell_N \geqslant 0$ span a dense subspace of $\mathcal{H}$. If x is cyclic in this sense, then the map

$$T : C(\sigma(A_1) \times \ldots \times \sigma(A_N)) \to \mathcal{H} : f \mapsto f(A_1, \ldots, A_N)x$$

can now be shown to extend to a unitary isomorphism from $L^2(\sigma(A_1) \times \ldots \times \sigma(A_N), \mu_x)$ to $\mathcal{H}$, and it identifies each of the operators A_j with the operation of multiplication of L^2-functions by the bounded functions $f_j(\lambda_1, \ldots, \lambda_N) := \lambda_j$. If there is no cyclic vector, then one can argue as in part (n) if $\mathcal{H}$ is separable (or using Zorn's lemma if $\mathcal{H}$ is not separable) that $\mathcal{H}$ splits into a direct sum of mutually orthogonal closed subspaces $\mathcal{H}_n \subset \mathcal{H}$ that are invariant under all the operators $A_1, \ldots, A_N$ and their adjoints, and on which they admit a common spectral representation as described above. This therefore identifies $\mathcal{H}$ with a direct sum of spaces $L^2(\sigma(A_1) \times \ldots \times \sigma(A_N), \mu_{x_n})$ with various spectral measures for cyclic vectors $x_n \in \mathcal{H}_n \subset \mathcal{H}$. This direct sum of L^2-spaces is naturally isomorphic to the space of L^2-functions on the disjoint union of all the measure spaces $(\sigma(A_1) \times \ldots \times \sigma(A_N), \mu_{x_n})$, and if $\mathcal{H}$ is separable so that there are at most countably many of these, we can rescale the vectors x_n in order to ensure without loss of generality that the total measure of this disjoint union is finite.

Solution to Problem 1

① (a) $A \in \mathcal{L}(\mathcal{H}) \geq 0 \quad (\langle Ax, x\rangle \geq 0 \;\; \forall x \in \mathcal{H})$

$\iff \sigma(A) \subset [0, \infty)$

$\Rightarrow$ $A \in \mathcal{L}(\mathcal{H})$ is s.a. and positive

$\Rightarrow$ $\sigma(A) \subseteq \mathbb{R}$

Let $\lambda \in (0, \infty)$. We'll prove that $-\lambda \notin \sigma(A)$

$-\lambda - A$ is invertible. (We'll prove)

Let $x \in \mathcal{H}$, then

$$\|(-\lambda - A)x\|^2 = \langle Ax + \lambda x, Ax + \lambda x\rangle$$
$$= \|Ax\|^2 + |\lambda|^2 \|x\|^2 + \langle \lambda x, Ax\rangle + \langle Ax, \lambda x\rangle$$

$$= \|Ax\|^2 + |\lambda|^2\|x\|^2 + 2\lambda\langle x, Ax\rangle$$
$$\geq |\lambda|^2\|x\|^2$$

Prob 4 of midterm $\Rightarrow$ $-\lambda - A$ is injective and has closed range.

$-\lambda - A$ is s.a. $\Rightarrow$ empty residual spectrum

$\Rightarrow$ $\operatorname{im}(-\lambda - A)$ is dense in $\mathcal{H}$

$\Rightarrow$ $-\lambda - A$ is also surjective

$\Rightarrow$ $-\lambda - A$ is bijective.

$\boxtimes$

Suppose $\sigma(A) \subseteq [0,\infty)$. We want to prove that A is positive, i.e. $A \geq 0$.

Let $\lambda = \inf\limits_{\|x\|=1} \langle Ax, x\rangle$ then we'll prove that

$\lambda \in \sigma(A)$. $\because$ $\sigma(A) \subseteq [0,\infty)$ this will imply that $A \geq 0$.

Consider the bilinear form

$T(u,v) = \langle Au - \lambda u, v\rangle$ which is symmetric and satisfies $T(v,v) \geq 0 \quad \forall v \in \mathcal{H}$.

$\Rightarrow$ T satisfies the Cauchy-Schwarz inequality and we get

$$|T(u,v)| \leq T(u,u)^{1/2}\, T(v,v)^{1/2} \quad \forall u,v \in \mathcal{H}$$

$\Rightarrow$

$$|Au - \lambda v| \leq \langle Au - \lambda u, u\rangle^{1/2} \langle Av - \lambda v, v\rangle^{1/2} \quad \forall u,v \in \mathcal{H}$$

$\Rightarrow \quad |Au - \lambda u| \leq C \langle Au - \lambda u, u\rangle^{1/2} \quad \forall u \in \mathcal{H}$ —— (1)

as $\lambda = \inf_{\|x\|=1} \langle Ax, x\rangle \Rightarrow \exists$ a sequence $(u_n) \in \mathcal{H}$

w/ $\|u_n\| = 1$ and $\langle Au_n, u_n\rangle \longrightarrow \lambda$. Eq. (1) $\Rightarrow$

$|Au_n - \lambda u_n| \to 0$ and i.e. $\lambda \in \sigma(A)$

or else if $\lambda \in \rho(A)$ then $u_n = (A - \lambda I)^{-1}(Au_n - \lambda I u_n)$

$\to 0$ which is impossible.

$\therefore \lambda \in \sigma(A)$ and hence $A \geq 0$.

(b) Let $\mathcal{H}$ be separable w/ (e_n) an o.n.b.

set $Ae_n = \frac{1}{n} e_n \implies \langle Au, u \rangle > 0 \quad \forall u \neq 0 \in \mathcal{H}$.

But of course $0 \in \sigma(A)$.

Solution to 2(a)

Spectral measure $\mu_x \rightsquigarrow A \in \mathcal{L}(\mathcal{H})$ (s.a.)

$x \in \mathcal{H}$ on $\sigma(A) \subseteq \mathbb{R}$ s.t

$$\langle x, f(A) x \rangle = \int_{\sigma(A)} f \, d\mu_x \quad \forall f \in C(\sigma(A)).$$

(a) Suppose $x \in \mathcal{H}$ is an eigenvector of A

$Ax = \lambda x, \quad \lambda \in \sigma(A)$

Claim:- $\mu_x = \|x\|^2 \delta_\lambda$ where δ_λ is the Dirac-delta measure centred at $\lambda \in \sigma(A)$.

$$\delta_\lambda(S) = \begin{cases} 1, & \lambda \in S \\ 0, & \lambda \notin S \end{cases}$$

If $f : \sigma(A) \to \mathbb{C}$

$$\int_{\sigma(A)} f \, d\delta_\lambda = f(\lambda)$$

$h \in C(\sigma(A))$ then

$$\langle x, f(A)x \rangle = \langle x, f(\lambda)x \rangle$$
$$= f(\lambda)\|x\|^2 = \|x\|^2 \int_{\sigma(A)} f \, d\delta_\lambda$$
$$= \int_{\sigma(A)} f \, d\mu_x .$$

$\boxed{}$

Solution to 3(a)

If T is closed then so in the operator

$D \to Y$

$x \mapsto Tx + Ax$, A is any bounded ,

$A \in \mathcal{L}(X, Y)$.

$(x, \text{im}(T+A)) \in D \times Y$ is Γ_{T+A}.

Suppose $(x_n, Tx_n + Ax_n)$ is a sequence

ie $D \times \Gamma_{T+A} \longrightarrow (x, y) \in X \times Y$.

$\Rightarrow \quad \|x - x_n\|_X + \|y - (Tx_n + Ax_n)\|_Y \longrightarrow 0$

$\Rightarrow \quad x_n \longrightarrow x \qquad \|y - Tx_n - Ax_n\| \to 0.$

We'd prove $y = Tx + Ax$.

$\because \|x - x_n\|_X \to 0 \Rightarrow \|Ax - Ax_n\|_Y \to 0$

A is continuous

$$\|y - Tx - Ax\| \leq \|y - Ax_n - Tx_n\| + \|Ax_n - Ax\|$$

$\Rightarrow$

$$\|y - Tx - Ax\| = 0$$

$\Rightarrow \quad y = Tx + Ax$

$\Rightarrow \quad \Gamma_{T+A}$ is closed. ($T+A$ is closed)

$\blacksquare$

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