Problem Session 3

Plan ① Differential under the integral sign

② Problem set 2.

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① Differential under the integral sign

Recall the Dominated Convergence Theorem

(X, μ) measure space w/

$f_n : X \longrightarrow \mathbb{R}$ measurable $\forall n \in \mathbb{N}$

$f : X \longrightarrow \mathbb{R}$ with $f_n(x) \xrightarrow{n \to \infty} f(x)$

for a.e. x.

Suppose $|f_n(x)| \leq g(x)$ for some $g \in \mathcal{L}^1(\mu)$

$\forall n$. Then $f_n \in \mathcal{L}^1(\mu)$, $f \in \mathcal{L}^1(\mu)$ and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Differential under the integral sign

Thm Suppose (Y, μ) is a measure space, M is a metric space and $\varphi : M \times Y \to \mathbb{R}$ is a function s.t.

1) $\forall x \in M$, $\varphi(x, \cdot) : Y \to \mathbb{R}$ is measurable and satisfies $|\varphi(x, \cdot)| \leq \psi$ for some $\psi \in \mathcal{L}^1(Y, \mu)$ independent of x.

2) $\forall y \in Y$, $\varphi(\cdot, y) : M \to \mathbb{R}$ is continuous.

Then $F : M \to \mathbb{R}$ given by

$$F(x) = \int_Y \varphi(x, \cdot)\, d\mu \text{ is continuous.}$$

If $M \underset{\text{open}}{\subset} \mathbb{R}^n$ w/ coordinates $x = (x_1, \ldots, x_n)$

and $\dfrac{\partial \varphi}{\partial x_j} : M \times Y \to \mathbb{R}$ exist for all $j = 1, \ldots, n$

and also satisfy the conditions above, the F is continuous differentiable and

$$\partial_j F(x) = \int_Y \frac{\partial \varphi}{\partial x_j}(x,\cdot)\, d\mu.$$

Sketch of proof cf. Thm 0.4 in lec. notes.

F is continuous

We'll prove $x_n \to x \Rightarrow F(x_n) \to F(x)$

$F(x_n) = \int \varphi(x_n,\cdot)\, d\mu$ $\qquad F(x) = \int \varphi(x,\cdot)\, d\mu$

Want $\int \varphi(x_n,\cdot)\, d\mu \xrightarrow{n\to\infty} \int \varphi(x,\cdot)\, d\mu$

$\varphi(\cdot, y)$ is a continuous function

$\Rightarrow$ $\varphi(x_n,\cdot) \longrightarrow \varphi(x,\cdot)$ pointwise.

$|\varphi(x_n,\cdot)| \leq \psi \in \mathcal{L}^1(\mu)$ (hypothesis)

DCT, $\int \varphi(x_n,\cdot)\, d\mu \longrightarrow \int \varphi(x,\cdot)\, d\mu$

$\Rightarrow$ F is continuous.

Hypothesis

$\frac{\partial \varphi}{\partial x_j}(x,y)$ exists, $\frac{\partial \varphi}{\partial x_j}(\cdot, y)$ is continuous $: M \to \mathbb{R}$

$\frac{\partial \varphi}{\partial x_j}(x,\cdot)$ is measurable

$\left|\frac{\partial \varphi}{\partial x_j}(x,\cdot)\right| \le \psi \in \mathcal{L}^1$ independent of x.

$\frac{\partial \varphi}{\partial x_j}(x,y)$ is

$\{e_1, e_2, \dots, e_n\}$ is a basis of $\mathbb{R}^n$.

$$\lim_{h\to 0} D_j^h \varphi(x,y) = \lim_{h\to 0} \frac{\varphi(x+he_j, y) - \varphi(x,y)}{h}$$

$\forall x \in M$, $D_j^h \varphi(x,\cdot) : Y \to \mathbb{R}$ is defined $\forall h \in \mathbb{R}\setminus\{0\}$.

If $h_n \to 0$, $h_n \in \mathbb{R}\setminus\{0\}$, then

$$\underbrace{D_j^{h_n} \varphi(x,\cdot)}_{} \longrightarrow \frac{\partial \varphi}{\partial x_j}(x,\cdot) \quad \text{pointwise on } Y.$$

write a formula for using the FTC.

$$D_j^h \varphi(x,y) = \int_0^1 \frac{\partial \varphi}{\partial x_j}(x+the_j, y)\, dt$$

$$|D_j^h \varphi(x,y)| \le \psi \Rightarrow \text{DCT}$$

$\boxed{\text{W}}$

ex. $$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

$$\int_0^{\infty} e^{-\frac{x^2}{2}}\, dx$$

We'll show $\displaystyle\int_0^\infty e^{-\frac{x^2}{2}}dx = \sqrt{\frac{\pi}{2}}$

For $t \in \mathbb{R}$

$$F(t) = \int_0^\infty \frac{e^{-\frac{t^2(x^2+1)}{2}}}{1+x^2}\,dx$$

$$F(0) = \int_0^\infty \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^\infty = \frac{\pi}{2}.$$

$$F(\infty) = 0$$

$$F'(t) = \int_0^\infty \frac{1}{\cancel{1+x^2}}\, e^{-\frac{t^2(x^2+1)}{2}} \left(-\frac{\cancel{2}t\,\cancel{(x^2+1)}}{\cancel{2}}\right) dx$$

$$= \int -t\, e^{-\frac{t^2(x^2+1)}{2}}\, dx$$

$$= -t\,e^{-\frac{t^2}{2}} \int e^{-\frac{(tx)^2}{2}}\, dx$$

put $tx = y$

$\Rightarrow t\,dx = dy$

$$F'(t) = -I\,e^{-t^2/2}$$

for any $b > 0$

$$\int_0^b F'(t)\,dt = -I \int_0^b e^{-\frac{t^2}{2}}\,dt$$

$$\Rightarrow \quad F(b) - F(0) = -I \int_0^b e^{-\frac{t^2}{2}} dt$$

let $b \to \infty$

$$\Rightarrow \quad F(\infty) - F(0) = -I \cdot I = -I^2$$

$$\Rightarrow \quad -\frac{\pi}{2} = -I^2 \Rightarrow I = \sqrt{\frac{\pi}{2}}$$

$\boxed{\square}$

PSET 2

(1) X, $V \subset X$ subspace, $\dim(X/V) = k$

(a) i) $\Rightarrow$ ii) $\dim(X/V) = 1$

Want $\forall x \in X$, $\exists! \, v \in V, \lambda \in \mathbb{K}$ s.t. $x = v + \lambda w$.
$\exists \, w \in X \setminus V$

Let $\{[w]\}$ is a basis of X/V.

Let $x \in X$. If $x \in V \Rightarrow x = x \in V$, done.

If $x \in X \setminus V \Rightarrow [x]$ is a non-trivial element in X/V

$\Rightarrow \quad [x] = \lambda [w] \quad$ for some $\lambda \in \mathbb{K}$

$X/V = \{[\alpha] + V \mid [\alpha] = \alpha + V\}$

$\Rightarrow \quad x - \lambda w \in V$

$\Rightarrow \quad x - \lambda w = v$ for some $v \in V$

$[0]$ element in $X/V = \{0+V\} = V$

$\Rightarrow \quad \boxed{x = v + \lambda w}$

<u>Uniqueness</u> Suppose $v' \in V$, $\lambda' \in \mathbb{K}$ s.t.

$$x = v' + \lambda' w$$

$\Rightarrow \quad v - v' = (\lambda - \lambda') w$

can only happen if $\lambda - \lambda' = 0 \Rightarrow \lambda = \lambda'$

$v = v'$.

$\square$

ii) $\Rightarrow$ iii) Want $V = \ker \Lambda$, $\Lambda : X \to \mathbb{K}$.

Let $x \in X$, then ii) $\Rightarrow$ $x = v + \lambda w$ for some $\lambda \in \mathbb{K}$, $v \in V$, $w \in X/V$.

Define

$$\Lambda : X \longrightarrow \mathbb{K} \text{ by}$$

Linear

$$\Lambda(x) = \lambda.$$

$\ker(\Lambda) = V$. $\quad$ <u>Λ is nontrivial.</u>

$\Lambda \not\equiv 0$. $w \in X \setminus V$ and $w = 0 + 1 \cdot w$

$\Lambda(w) = 1. \neq 0. \Rightarrow \Lambda \not\equiv 0.$

iii) $\Rightarrow$ i) $\exists\ \Lambda \not\equiv 0$, s.t. $V = \ker \Lambda$.

$\exists\ x_0 \in X$ s.t. $\Lambda(x_0) \neq 0$. $x_0 \notin V$ b/c $V = \ker \Lambda$

$\Rightarrow \Lambda(x_0) = 0$ ✗

$\Rightarrow x_0 \in X \setminus V$.

Want :- V is codim 1 $\Leftrightarrow \frac{X}{V}$ is 1-dimensional.

Claim :- $\{[x_0]\}$ is a basis

Suppose $[\alpha] \in X/V \Rightarrow \alpha = y + V$ for some $y \in X$.

Then $$\Lambda\left(y - \frac{\Lambda(y)}{\Lambda(x_0)} x_0\right) = 0$$

$$\Rightarrow y - \frac{\Lambda(y)}{\Lambda(x_0)} x_0 \in \ker \Lambda = V$$

$$\Rightarrow y = \frac{\Lambda(y)}{\Lambda(x_0)} x_0 + v, \quad v \in V$$

$$\Rightarrow \alpha = \frac{\Lambda(y)}{\Lambda(x_0)} x_0 + V$$

$$\Rightarrow [\alpha] = c[x_0], \quad \text{where } c = \frac{\Lambda(y)}{\Lambda(x_0)}.$$

1\. c) b) $\Rightarrow$ c)

If V is not dense $\Rightarrow$ $\Lambda : X \to \mathbb{K}$ is continuous.

If $\Lambda : X \to \mathbb{K}$ is NOT continuous $\Rightarrow$ $V = \ker \Lambda$ is dense in X.

Λ is not bounded

$\Rightarrow$ $\forall n \in \mathbb{N}$, $\exists$ $x_n \in X$ s.t. $\Lambda(x_n) \geq n \, \|x_n\|$

rescale, s.t. $\|x_n\| = 1$

$\Rightarrow$ $\Lambda(x_n) \geq n$

Let $y \in X$ arbitrary.

Then $y_n = y - \dfrac{\Lambda(y)}{\Lambda(x_n)} x_n$, $\boxed{y_n \to y}$

$$\Lambda(y_n) = \Lambda(y) - \Lambda(y) = 0$$

$\Rightarrow$ $\{y_n\}$ is a sequence in V.

$y_n \longrightarrow y$ $\Rightarrow$ V is dense in X.

2\) b) L^∞, $f(x) = x$ not strictly convex
$g(x) = x^2$

L^1 , $f = 1$, $g(x) = 2x$

(4) $L^p([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid$

$\left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} < \infty \}$

$0 < p < 1$.

$d(f,g) = \|f-g\|_{L^p}^p$

a) for $a, b \geq 0$ and $q \geq 1$

$a^q + b^q \leq (a+b)^q$. ✓

$\frac{a}{a+b} \leq 1 \Rightarrow \left(\frac{a}{a+b} \right)^q \leq \frac{a}{a+b}$

$\left(\frac{b}{a+b} \right)^q \leq \frac{b}{a+b}$

$\Rightarrow \frac{a^q}{(a+b)^q} + \frac{b^q}{(a+b)^q} \leq 1$

$\Rightarrow a^q + b^q \leq (a+b)^q$

b) $+$ continuous

and $\cdot$ continuous

$$f_n \to f \qquad \longrightarrow \quad f_n + g_n \to f+g$$
$$g_n \to g$$

triangle inequality.

c) $\{ |f| \geq M \ \forall \ M \in \mathbb{R} \}$ is of measure zero.

$$A_n = \{ x \in [0,1] \, | \, |f(x)| \leq n \}_{n \in \mathbb{N}}$$

$$f_n = f \chi_{A_n}$$

$$d(f_n, f) \longrightarrow 0 \qquad \int_0^1 \| f_n - f \|^p \, d\mu \to 0$$

$$f \in L^p([0,1]) \quad , \quad \int_0^1 |f(x)|^p \, d\mu < \infty .$$

d) for $\tau_1 \, \tau_2$, x_1, x_2 — convex set.

$$\{x_1, \dots, x_{n-1}\} , \{\tau_1, \tau_2, \dots, \tau_{n-1}\}$$

$$y = \frac{\tau_1}{\tau_1 + \dots + \tau_{n-1}} x_1 + \dots + \frac{\tau_{n-1}}{\tau_1 + \tau_2 + \dots \tau_{n-1}} x_{n-1}$$

$\in K$ (induction hypo)

$\to y(\tau_1 + \dots + \tau_{n-1}) + \tau_n x_n \in K.$

e) Let $\epsilon > 0$ and $N \geq 1$

$[0, 1]$

Choose N disjoint intervals in $[0,1]$ say $I_1, I_2, \dots, I_n$.

$$f_K = \frac{\left(\frac{\epsilon}{2}\right)^q}{\ell(I_K)^q} \chi_{I_K} \quad : [0,1] \to \mathbb{R}$$

where $q = \frac{1}{p}$.

$q > 1$

Then

$$\int_0^1 |f_K(x)|^p dx = \int_0^1 \left[\frac{\left(\frac{\epsilon}{2}\right)^{1/p}}{(\ell(I_K))^{1/p}} \chi_{I_K}\right]^p dx$$

$$= \frac{\epsilon}{2}$$

$\therefore \quad d(f_K, 0) = \|f_K\|_{L^p}^p = \frac{\epsilon}{2}$

$\therefore \quad d(f_K, 0) < \epsilon.$ —— $(*)$

$$g_N = \frac{1}{N}\sum_{k=1}^{N} f_k$$

$$\int_0^1 |g_n(x)|^p dx = \frac{1}{N^p}\sum_{k=1}^{N}\int_0^1 |f_k(x)|^p dx$$

$$= \frac{N\epsilon}{2N^p} = \frac{N^{(1-p)}\epsilon}{2}$$

$$1-p > 0$$

choosing N large enough $\Rightarrow$

$$d(g_N, 0) > \epsilon.$$

$\therefore$ $g_n \notin B_\epsilon(0)$.

$\Rightarrow$ $L^p([0,1])$ is not locally convex.

f) Suppose not.

Let $\Lambda \in L^p[0,1]^*$ s.t. $\Lambda \neq 0$

$\Rightarrow$ $\Lambda : L^p([0,1]) \to \mathbb{R}$

and $\operatorname{Im}(\Lambda) = \mathbb{R}$.

$\Rightarrow \exists\, f \in L^p([0,1])$ with $|\Lambda(f)| \geq 1$.

Want to contradict that Λ is continuous.

We'll $\{h_n\}$ s.t. $h_n \to 0$, $\Lambda(g_n) \not\to 0$.

$$[0,1] \longrightarrow \mathbb{R} \quad \text{by}$$
$$s \longmapsto \int_0^s |f(x)|^p \, dx$$

continuous function on $[0,1] \Rightarrow \exists\ s \in (0,1)$

s.t.
$$\int_0^s |f(x)|^p dx = \frac{1}{2} \int_0^1 |f(x)|^p dx > 0$$
(IVT)

$$g_1 = f \chi_{[0,s]} \text{ and } g_2 = f \chi_{(s,1]}$$

$$f = g_1 + g_2$$

and $|f|^p = |g_1|^p + |g_2|^p$

$$\int_0^1 |g_1(x)|^p dx = \int_0^s |f(x)|^p dx = \frac{1}{2}\int_0^1 |f(x)|^p dx$$

$$\therefore \int_0^1 |g_2(x)|^p dx = \frac{1}{2}\int_0^1 |f(x)|^p dx$$

$$\because |\Lambda(f)| \geq 1 \Rightarrow |\Lambda(g_1+g_2)| \geq 1$$

$$\Rightarrow |\Lambda(g_1)+\Lambda(g_2)| \geq 1$$

$\Rightarrow$ $i = 1$ or 2 $|\Lambda(g_i)| \geq \frac{1}{2}$.

Let $h_1 = 2g_i \Rightarrow |\Lambda(h_1)| \geq 1$

and
$$\int_0^1 |h_1(x)|^p dx = 2^p \int_0^1 |g_i(x)|^p dx$$
$$= 2^p \cdot \frac{1}{2}\int_0^1 |f(x)|^p dx$$
$$= \underbrace{2^{p-1}}_{<1} \int_0^1 |f(x)|^p dx \quad , p \in (0,1)$$

Keep iterating to get a sequence $\{h_n\}$.

$|\Lambda(h_n)| \not\to 0$ but $d(h_n, 0) \to 0$

contradiction Λ is continuous.

$\Lambda \equiv 0$.

$\square$